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# Behavior of the Parallel Code Method

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**ABSTRACT**

Parallel Code Method (PCM) is one of iterative methods for solving nonlinear equations. The method uses an initial constant matrix instead of Jacobi matrix at every iteration. First, the notion of local convergence for PCM is defined and a sufficient condition for local convergence is given. Secondly, it is shown that the approximations generated by PCM are represented by an infinite power series under the condition that a root of the nonlinear equation is not multiple. Next, based on this property, the sequence of approximations of PCM is accelerated and a better approximation of a root is estimated. Finally, it is verified by numerical examples that the sequence is accelerated according to the property.

**Keywords:** Numerical Analysis, Nonlinear Equation, Iterative Method, Acceleration, Newton-Raphson's Method

## 1. Introduction

It is difficult to solve nonlinear equations analytically because they have complicated structure in general. Therefore, various iterative methods have been proposed to solve them. For example, Newton-Raphson's method, which is frequently used to solve nonlinear equations, has a quadratic convergence [1, 2]. The reason is that the method uses a differential information, which is a Jacobian matrix, at the newest approximation. However, it takes many calculations and long time to get a Jacobian matrix at every iteration. On the other hand, we treat Parallel Code Method (PCM) [3] or known to von Muses Method [4]. The method is an iterative method for solving a nonlinear equation of one variable by using an initial constant matrix instead of a Jacobi matrix at every iteration. The method demands little calculation at every iteration but the convergence is linear and slow. However, we find a remarkable property of PCM that the sequence of approximations is represented by an infinite power series. Based on this property, we propose an acceleration method of the sequence of approximations generated by PCM. By numerical examples, we verify that the sequence is accelerated according to the property.

## 2. Local Convergence

Let $f(x)$ be a continuous differential nonlinear function of one variable. We shall find a root of the equation $f(x) = 0$. Now, let $x_0$ be an initial approximation of a root. By using the iterative method:

$$x_{n+1} = x_n - \varphi(x_n) f(x_n), \tag{1}$$

the approximation $x_n$ is refined. Here, the function $\varphi(x)$ is continuous. If $\varphi(x_n) = 0$ for some $n$, the ap-

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proximation $x_n$ is not influenced by the iteration (1). So, we demand that the function $\varphi(x_n)$ is not equal to zero for any integers $n \geq 0$. If a sequence $\{x_n\}$ converges, we have

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} \frac{x_n - x_{n+1}}{\varphi(x_n)} = 0. \quad (2)$$

Therefore, the sequence $\{x_n\}$ converges to a root of the equation $f(x) = 0$.

**Definition 2.1** PCM has a *local convergence* at a root $p$ if there exists a positive M such that $\{x_n\}$ converges to $p$ for any $x_0 \in [p - M, p + M]$.

A sufficient condition for a function $\varphi(x)$ where the iterative method (1) has a local convergence at a root $p$ is given in the following theorem.

**Theorem 2.1** If the inequality

$$|1 - \varphi(p)f'(p)| \quad < \quad 1 \quad (3)$$

holds, the sequence $\{x_n\}$ of the iterative method (1) locally converges to a root $p$ of the equation $f(x) = 0$.

**Proof:** From the differentiability of the function $f(x)$, it can be expanded in a Taylor series as

$$f(x) \quad = \quad f'(p)x + o(x). \quad (4)$$

Substituting (4) into (1), we get

$$\frac{x_{n+1}}{x_n} \quad = \quad 1 - \varphi(x_n)f'(p) + \varphi(x_n) \cdot \frac{o(x_n)}{x_n}. \quad (5)$$

Using the Contraction Principle, the sequence $\{x_n\}$ converges locally under the condition of the inequality (3). ■

## 3. Properties

*Parallel Code Method* is defined by the iteration:

$$x_{n+1} \quad = \quad x_n - \mu f(x_n), \quad (6)$$

where $\mu$ is a suitable constant. It is the special case that the function $\varphi(x)$ is a constant $\mu$ in the iteration (1). Behavior of the sequence of PCM that converges to a root of a equation do not depend on the coordinates. Hence, to simplify discussion, we introduce a new sequence $y_n = x_n - p$ and a new function $g(x) = f(x+p)$. Then PCM is represented by

$$y_{n+1} \quad = \quad y_n - \mu g(y_n) \quad (7)$$

and we shall find the root 0 of the nonlinear equation $g(x) = 0$. [See Fig. 1]

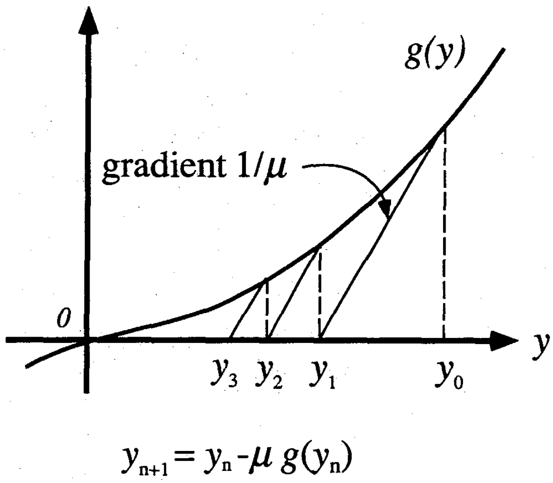


Fig1. Parallel Code Method.

Here after, we assume that a constant $\mu$ satisfies the sufficient condition of Theorem 2.1:

$$|1 - \mu g'(0)| \quad < \quad 1. \quad (8)$$

This condition also means that the root 0 is not multiple since the differential coefficient $g'(0)$ is not equal to zero.

**Lemma 3.1** If there holds $\mu \neq \frac{1}{g'(0)}$, PCM has a linear convergence. In other words,

$$\lim_{n\to\infty} \left| \frac{y_{n+1}}{y_n} \right| = |1 - \mu g'(0)| \quad (9)$$

holds.

**Proof:** From (5), we have

$$\frac{y_{n+1}}{y_n} \quad = \quad 1 - \mu g'(0) + \frac{o(y_n)}{y_n}. \quad (10)$$

Limiting $n$ to infinity, we have the equation (9). ■

We assumed that a nonlinear function $f(x)$ is continuous and differential. Moreover, we assume that the function $f(x)$ is regular in a neighborhood of a root $p$. Then the function $g(x) = f(x+p)$ can be represented by a power series as

$$g(x) = \sum_{k=1}^{\infty} a_k x^k \tag{11}$$

at the root 0. Let use the notation $\alpha \triangleq 1 - \mu a_1$. Since $a_1 = g'(0)$, we get

$$\begin{aligned} y_{n+1} &= y_n - \mu \sum_{k=1}^{\infty} a_k y_n^k \\ &= \alpha y_n \left\{ 1 - \frac{\mu}{\alpha} \sum_{k=1}^{\infty} a_{k+1} y_n^k \right\} \end{aligned} \tag{12}$$

from (7). From this recurrent formula, we have

$$y_n = \alpha^n y_0 \prod_{i=0}^{n-1} \left\{ 1 - \frac{\mu}{\alpha} \sum_{k=1}^{\infty} a_{k+1} y_i^k \right\}. \tag{13}$$

In this time, the following lemma holds.

**Lemma 3.2** The infinite product

$$y_0 \prod_{i=0}^{\infty} \left\{ 1 - \frac{\mu}{\alpha} \sum_{k=1}^{\infty} a_{k+1} y_i^k \right\} \tag{14}$$

converges if the sequence $\{y_n\}$ converges locally. Therefore, let $M$ denote the value of (14) then we get

$$\lim_{n\to\infty} \frac{y_n}{\alpha^n} = M. \tag{15}$$

**Proof:** It is well known that the infinite product (14) converges if the double sequence

$$\sum_{i=0}^{\infty} \sum_{k=1}^{\infty} a_{k+1} y_i^k \tag{16}$$

converges absolutely [5]. From Lemma 3.1, we have

$$\lim_{n\to\infty} \left| \frac{y_{n+1}}{y_n} \right| = |\alpha| < 1. \tag{17}$$

Therefore, if $\beta$ satisfies $0 < |\alpha| < \beta < 1$, there exists a large number $N$ such that

$$|y_{n+1}| < \beta |y_n| \tag{18}$$

for all $n \geq N$. Let separate

$$\begin{aligned} &\sum_{i=0}^{\infty} \sum_{k=1}^{\infty} |a_{k+1} y_i^k| \\ &= \sum_{i=0}^{N-1} \sum_{k=1}^{\infty} |a_{k+1} y_i^k| + \sum_{i=N}^{\infty} \sum_{k=1}^{\infty} |a_{k+1} y_i^k|. \end{aligned} \tag{19}$$

Since approximations $y_n$ $(n = 0, 1, 2, \cdots)$ are included in the convergent radius of the power series (11), the first part

$$\sum_{i=0}^{N-1} \sum_{k=1}^{\infty} |a_{k+1} y_i^k| \tag{20}$$

converges. On the other hand, from (18), it holds $|y_{N+n}| < \beta^n |y_N|$. Therefore we have

$$\begin{aligned} \sum_{k=1}^{\infty} \sum_{i=N}^{\infty} |a_{k+1} y_i^k| &\leq \sum_{k=1}^{\infty} |a_{k+1}| \sum_{i=0}^{\infty} \beta^{ki} |y_N|^k \\ &= \sum_{k=1}^{\infty} |a_{k+1}| \frac{|y_N|^k}{1-\beta^k} \\ &\leq \frac{1}{1-\beta} \sum_{k=1}^{\infty} |a_{k+1}| |y_N|^k. \end{aligned} \tag{21}$$

Since $y_N$ is also included in the convergent radius of $g(x)$, the second part also converges. Hence, the infinite product (16) converges absolutely and infinite product also converges. ■

This lemma means that the approximations $y_n$ are approximated as

$$y_n \approx M\alpha^n \tag{22}$$

when $n$ is sufficient large. In other words, the function $g(y)$ can be approximated by a linear function $g'(0)y$ in a neighborhood of the root 0 and the sequence $\{y_n\}$ can be approximated by a geometric series $\{M\alpha^n\}$ with the convergent rate $\alpha = 1 - \mu g'(0)$. [See Fig. 2]

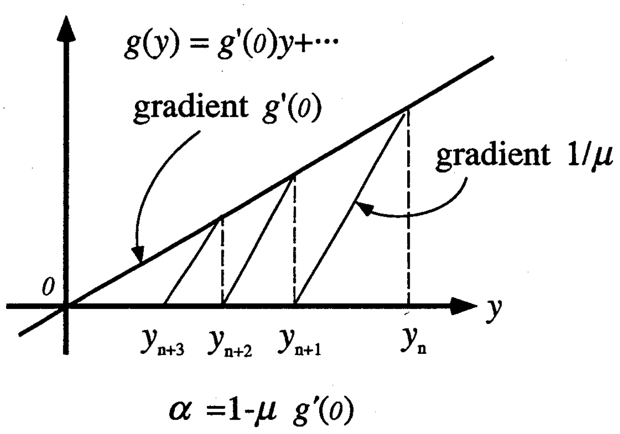


Fig2. Behavior of the sequence of PCM.

The approximate expression (22) can be extended to the following theorem.

**Theorem 3.2** Let a regular function $g(x)=\sum_{i=1}^{\infty} a_i x^i$ has a nonzero convergent radius. Let $\{y_n\}$ be the sequence of the approximations of PCM applied to the function $g(x)$. Then, there exist constants $A_i$ for $i \geq 1$ and a sufficient large integer $N$ such that approximations are represented by a power series as

$$y_n = \sum_{i=1}^{\infty} A_i \alpha^{in} \tag{23}$$

for all $n \geq N$. Here, constants $\{A_i\}$ are obtained by a recurrent formula:

$$A_1 = M \tag{24}$$

$$A_i = \frac{1-\alpha}{a_1\alpha(1-\alpha^{i-1})} \sum_{j=2}^{i} a_j \prod_{\substack{l=1,1\leq k_l\leq j \\ k_1+\cdots+k_j=i}}^{j} A_{k_l} \quad (i \geq 2). \tag{25}$$

Before proving Theorem 3.2, we prepare two lemmas.

**Lemma 3.3** Each $A_i$ $(i \geq 1)$ defined by the recurrent formula (25) is represented by

$$A_i = c_i A_1^i, \tag{26}$$

where constants $c_i$ are independent of the value of $A_1$.

**Proof:** We prove this by using induction for $i$.
(1) When $i = 1$, it is obvious because $A_1 = A_1$ and $c_1 = 1$.
(2) When $i \leq k$, suppose that $A_i = c_i A_1^i$ for some constants $c_i$.
(3) When $i = k+1$, from (25), it holds

$$\begin{aligned} A_{k+1} &= \frac{1-\alpha}{a_1\alpha(1-\alpha^k)} \sum_{j=2}^{k+1} a_j \prod_{\substack{l=1,1\leq k_l\leq j \\ k_1+\cdots+k_j=k+1}}^{j} A_{k_l} \\ &= \frac{1-\alpha}{a_1\alpha(1-\alpha^k)} \sum_{j=2}^{k+1} a_j \prod_{\substack{l=1,1\leq k_l\leq j \\ k_1+\cdots+k_j=k+1}}^{j} c_{k_l} A_1^{k_l} \\ &= \frac{1-\alpha}{a_1\alpha(1-\alpha^k)} \sum_{j=2}^{k+1} a_j \left(\prod_{l=1}^{j} c_{k_l}\right) A_1^{k_1+\cdots+k_j} \\ &= \frac{1-\alpha}{a_1\alpha(1-\alpha^k)} \sum_{j=2}^{k+1} a_j \left(\prod_{\substack{l=1,1\leq k_l\leq j \\ k_1+\cdots+k_j=k+1}}^{j} c_{k_l}\right) A_1^{k+1}. \end{aligned} \tag{27}$$

Therefore we have $A_{k+1} = c_{k+1} A_1^{k+1}$. Consequently, $A_i$ for $i \geq 1$ are represented by the $i$th power of $A_1$. ■

**Lemma 3.4** If a regular function $g(x) = \sum_{i=1}^{\infty} a_i x^i$ has a nonzero convergent radius, a power series

$$G(x) \triangleq \sum_{i=1}^{\infty} c_i x^i \tag{28}$$

also has a nonzero convergent radius.

**Proof:** If $A_1 \neq 0$, it holds

$$G(A_1 x) = \sum_{i=1}^{\infty} c_i A_1^i x^i = \sum_{i=1}^{\infty} A_i x^i. \tag{29}$$

Therefore, if we prove that a power series $\sum_{i=1}^{\infty} A_i x^i$ has a nonzero convergent radius, a power series $G(x)$ also has a nonzero convergent radius. We define $B_i$ by the recurrent formula:

$$B_1 \triangleq |A_1| \tag{30}$$

$$B_i \triangleq \frac{1}{|a_1|(|\alpha| - |\alpha|^2)} \sum_{j=2}^{i} |a_j| \prod_{\substack{l=1,1\leq k_l\leq j \\ k_1+\cdots+k_j=i}}^{j} B_{k_l} \quad (i \geq 2). \tag{31}$$

Then, since $|\alpha| < 1$ and from the recurrent formula (25), we get $|A_i| \leq B_i$. Hence, if $H(x) \triangleq \sum_{i=1}^{\infty} B_i x^i$ has a nonzero convergent radius, a power series $\sum_{i=1}^{\infty} A_i x^i$ also has a nonzero convergent radius. Let introduce a new function:

$$h(x) \triangleq \frac{x}{|A_1|} - \frac{\sum_{i=2}^{\infty} |a_i| x^i}{|A_1 a_1|(|\alpha| - |\alpha|^2)}. \tag{32}$$

Since a regular function $g(x) = \sum_{i=1}^{\infty} a_i x^i$ has a nonzero convergent radius, the function $h(x)$ also has a nonzero convergent radius. Moreover, since it holds

$$h'(0) = \frac{1}{|A_1|} \neq 0, \tag{33}$$

from the Inverse Function Theorem, $h(x)$ has a unique inverse function in a neighborhood of $x = 0$ [7]. Here, since

$$h \circ H(x) = x, \tag{34}$$

the function $H(x)$ becomes the inverse function of $h(x)$ in a neighborhood $x = 0$. Therefore the power series $H(x)$ is regular in a neighborhood of $x = 0$ and has a nonzero convergent radius. Hence, the power series $G(x)$ also has a nonzero convergent radius. ■

**Proof of Theorem 3.2:** From Lemma 3.4, the function $G(x)$ has a nonzero convergent radius. Moreover, since $G(0) = 0$ and $G'(0) = c_1 = 1 > 0$, there exists a small number $m > 0$ such that $m$ is included in the convergent radius of $G(x)$, $G(m) > 0$ and $G(-m) < 0$. Especially, we determine this $m$ so small that $\sum_{i=1}^{\infty} |c_i| m^i$ is included in the convergent radius of the regular function $g(x) = \sum_{k=1}^{\infty} a_k x^k$. Here, since the limit of $y_n$ is equal to zero, there exists a large number $N$ such that $G(-m) < y_n < G(m)$ for all $n \geq N$. Furthermore, since $G(x)$ is continuous and from the Intermediate Value Theorem, there exists a point $m_0$ ($|m_0| < m$) such that

$$y_N = G(m_0) = \sum_{i=1}^{\infty} c_i m_0^i. \quad (35)$$

Let determine $A_1 = \frac{m_0}{\alpha^N}$ and using the recurrent formula (25), we determine $A_i (i \geq 2)$ one by one. Then, it follows Lemma 3.3 that

$$y_N = \sum_{i=1}^{\infty} c_i m_0^i = \sum_{i=1}^{\infty} c_i A_1^i \alpha^{iN} = \sum_{i=1}^{\infty} A_i \alpha^{iN}. \quad (36)$$

Next, we shall prove that

$$y_{N+k} = \sum_{i=1}^{\infty} A_i \alpha^{i(N+k)} \quad (37)$$

for $k \geq 0$ by using the induction for $k$.

(1) When $k = 0$, it holds from (36).

(2) When $k = n$, we assume

$$y_{N+n} = \sum_{i=1}^{\infty} A_i \alpha^{i(N+n)}. \quad (38)$$

(3) When $k = n + 1$, since $\mu = \frac{1-\alpha}{a_1}$, we have

$$\begin{aligned} & y_{N+n+1} \\ &= y_{N+n} - \mu g(y_{N+n}) \\ &= \sum_{i=1}^{\infty} A_i \alpha^{i(N+n)} - \mu \sum_{k=1}^{\infty} a_k \Big( \sum_{i=1}^{\infty} A_i \alpha^{i(N+n)} \Big)^k \\ &= \alpha \cdot \sum_{i=1}^{\infty} A_i \alpha^{i(N+n)} - \frac{1-\alpha}{a_1} \sum_{k=2}^{\infty} a_k \Big( \sum_{i=1}^{\infty} A_i \alpha^{i(N+n)} \Big)^k. \quad (39) \end{aligned}$$

Here, since $0 < |\alpha| < 1$, we get

$$\begin{aligned} & \sum_{k=1}^{\infty} |a_k| \Big( \sum_{i=1}^{\infty} |A_i \alpha^{i(N+n)}| \Big)^k \\ &= \sum_{k=1}^{\infty} |a_k| \Big( \sum_{i=1}^{\infty} |c_i| |A_1 \alpha^N|^i |\alpha|^{in} \Big)^k \\ &\leq \sum_{k=1}^{\infty} |a_k| \Big( \sum_{i=1}^{\infty} |c_i| |m_0|^i \Big)^k \\ &\leq \sum_{k=1}^{\infty} |a_k| \Big( \sum_{i=1}^{\infty} |c_i| m^i \Big)^k. \quad (40) \end{aligned}$$

Here, since $\sum_{i=1}^{\infty} |c_i| m^i$ is included in the convergent radius of $g(x) = \sum_{k=1}^{\infty} a_k x^k$, the double series

$$\sum_{k=1}^{\infty} a_k \Big( \sum_{i=1}^{\infty} A_i \alpha^{i(N+n)} \Big)^k \quad (41)$$

converges absolutely Therefore, we can change the order of the double summation arbitrary. Hence, from (39) and (25), we have

$$\begin{aligned} & y_{N+n+1} \\ &= \sum_{i=1}^{\infty} \Big\{ \alpha A_i - \frac{1-\alpha}{a_1} \sum_{j=2}^{i} a_j \prod_{\substack{l=1, 1 \leq k_l \leq j \\ k_1 + \cdots + k_j = i}}^{j} A_{k_l} \Big\} \alpha^{i(N+n)} \\ &= \sum_{i=1}^{\infty} A_i \alpha^{i(N+n+1)}. \quad (42) \end{aligned}$$

Finally, since $|\alpha| < 1$ and $y_n = \sum_{i=1}^{\infty} A_i \alpha^{in}$ for $n \geq N$, it holds

$$\begin{aligned} \lim_{n \to \infty} \frac{y_n}{\alpha^n} &= A_1 + \lim_{n \to \infty} \sum_{i=2}^{\infty} A_i \alpha^{(i-1)n} \\ &= A_1. \end{aligned} \quad (43)$$

Hence, from (15), we get $A_1 = M$. ■

In this theorem, the condition $n \geq N$ can not be replaced to $n \geq 0$. Because it happens that $\sum_{i=1}^{\infty} A_i = \infty$. However, if an initial approximation $y_0$ is sufficient near to a root 0, there holds

$$y_n = \sum_{i=1}^{\infty} A_i \quad (44)$$

for all $n \geq 0$. [See Ex.1 in Section 5.]

# 4. Acceleration

From Theorem 3.2 and since $x_n = y_n + p$, when $n$ is sufficient large, approximations $x_n$ are represented by

$$x_n = p + \sum_{i=1}^{\infty} A_i \alpha^{in}, \quad (45)$$

where $p$ is a single root of the function $f(x)$. It is known that a method for the Limit Estimation [8] or $\varepsilon$ algorithm [9, 10] are useful to accelerating the sequence of this type. A method for Limit Estimation accelerates a sequence $\{x_n\}$ by making a new sequence $\{p_n^{(m)}\}$:

$$p_n^{(m)} \triangleq \frac{\sum_{i=0}^{m} \lambda_i^{(n)} x_{n-i}}{\sum_{i=0}^{m} \lambda_i^{(n)}} \quad (n \geq 2m). \quad (46)$$

Here $\lambda_i^{(n)}$ for $i = 0, 1, 2, \cdots, m$ are determined by $\lambda_0^{(n)} = 1$ and simultaneous linear equations

$$\begin{pmatrix} \nabla x_{n-2m+1} \nabla x_{n-2m+2} \cdots \nabla x_{n-m} \\ \nabla x_{n-2m+2} \nabla x_{n-2m+3} \cdots \nabla x_{n-m+1} \\ \cdots \quad \cdots \quad \cdots \\ \nabla x_{n-m} \quad \nabla x_{n-m+1} \cdots \nabla x_{n-1} \end{pmatrix} \begin{pmatrix} \lambda_m^{(n)} \\ \lambda_{m-1}^{(n)} \\ \cdots \\ \lambda_1^{(n)} \end{pmatrix} = - \begin{pmatrix} \nabla x_{n-m+1} \\ \nabla x_{n-m+2} \\ \cdots \\ \nabla x_n \end{pmatrix}, \quad (47)$$

where $\nabla x_n \triangleq x_n - x_{n-1}$. The convergent rate of an original sequence $\{x_n\}$ is equal to $|\alpha|$. However, the rates of new sequences $\{p_n^{(m)}\}$ become $|\alpha|^{m+1}$ for $m \geq 1$. From the condition of local convergence, $|\alpha| < 1$ holds. Therefore the convergent rates are improved.

## 5. Examples

In order to verify above theorems, we practice some examples.

**[Ex. 1]** Let a nonlinear function $f(x)$ be

$$f(x) \quad = \quad x + x^2. \quad (48)$$

We shall find the root 0 of the equation $f(x) = 0$ by using PCM. Let an initial approximation $x_0 = 1$ and a constant $\mu = 0.2$. Then it holds

$$|\alpha| \quad = \quad |1 - 0.2f'(0)| = |1 - 0.2| = 0.8 < 1 \quad (49)$$

since $\alpha = 1 - \mu a_1$. Therefore this constant $\mu$ satisfies the condition of local convergence of Theorem 2.1.

In the beginning, we find the value of $A_1$. Since $a_1 = a_2 = 1$ and $a_i = 0$ for $i \geq 3$, the infinite product (14) becomes

$$A_1 \quad = \quad x_0 \prod_{i=0}^{\infty} \left\{ 1 - \frac{\mu x_i}{\alpha} \right\}. \quad (50)$$

If we omit to multiply more than 200 terms in the infinite product (14), we have

$$A_1 \approx x_0 \prod_{i=0}^{200} \left\{ 1 - \frac{\mu x_i}{\alpha} \right\} = 0.41460051 \cdots. \quad (51)$$

Fig. 3 shows the graph of $\sum_{i=1}^{n} A_i$ for $n = 1, 2, \cdots, 100$ which is determined by the recurrent formula (25). From this graph, it is seen that

$$\sum_{i=1}^{100} A_i \approx \sum_{i=1}^{\infty} A_i = x_0 = 1. \quad (52)$$

Therefore, when an initial approximation $x_0$ is sufficient near to a root, Theorem 3.2 holds for all $n \geq 0$. Table 1

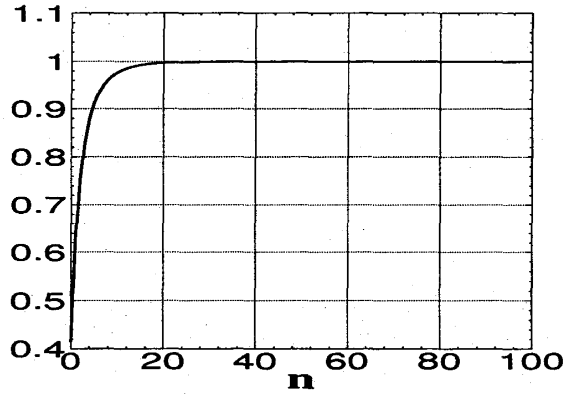


Fig3. Sum of $A_i$, $i$ equals one to $n$.

shows comparison between the original sequence $\{x_n\}$. and sequences $\{p_n^{(m)}\}$ $m = 1, 2, 3$ accelerated by the method for Limit Estimation. We understand that accelerated sequences $\{p_n^{(m)}\}$ converge faster if $m$ is larger. Table 2 shows the convergent rates of the sequences $\{x_n\}$ and $\{p_n^{(m)}\}$ $(m = 1, 2, 3)$. From this table, we can presume that each rate is equal to

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} \quad = \quad \alpha = 0.8 \quad (53)$$

$$\lim_{n\to\infty} \frac{p_{n+1}^{(m)}}{p_n^{(m)}} \quad = \quad \alpha^{m+1} = (0.8)^{m+1}, \quad (54)$$

respectively. Therefore, we can confirm that the sequence $\{x_n\}$ is represented by the power series as (45).

**[Ex. 2]** Let a nonlinear function $f(x)$ be

$$f(x) = \sin(x). \quad (55)$$

We shall find a root 0 of the equation $f(x) = 0$ by using PCM. Let $x_0 = 1$ and $\mu = 0.2$. Then it holds

$$|\alpha| \quad = \quad |1 - 0.2f'(0)| = |1 - 0.2| = 0.8 < 1. \quad (56)$$

Table 1. Comparison between $x_n$ and $p_n^{(m)}$.

| $n$ | $x_n$ | $p_n^{(1)}$ | $p_n^{(2)}$ | $p_n^{(3)}$ |
|---|---|---|---|---|
| 0 | 1.00000000 | | | |
| 1 | 0.60000000 | | | |
| 2 | 0.40800000 | 0.23076924 | | |
| 3 | 0.29310720 | 0.12191235 | | |
| 4 | 0.21730339 | 0.07029992 | 0.04883564 | |
| 5 | 0.16439856 | 0.04216946 | 0.02190101 | |
| 6 | 0.12611347 | 0.02585532 | 0.01038088 | 0.00768408 |
| 7 | 0.09770986 | 0.01606565 | 0.00505224 | 0.00282342 |
| 8 | 0.07625844 | 0.01006889 | 0.00249634 | 0.00107867 |
| 9 | 0.05984368 | 0.00634702 | 0.00124552 | 0.00042147 |
| 10 | 0.04715869 | 0.00401688 | 0.00062553 | 0.00016641 |
| $\infty$ | 0 | 0 | 0 | 0 |

Table 2. Convergent rates of $x_n$ and $p_n^{(m)}$.

| $n$ | $\frac{x_{n+1}}{x_n}$ | $\frac{p_{n+1}^{(1)}}{p_n^{(1)}}$ | $\frac{p_{n+1}^{(2)}}{p_n^{(2)}}$ | $\frac{p_{n+1}^{(3)}}{p_n^{(3)}}$ |
|---|---|---|---|---|
| 0 | 0.60000000 | | | |
| 1 | 0.68000000 | | | |
| 2 | 0.71840000 | 0.52828685 | | |
| 3 | 0.74137856 | 0.57664312 | | |
| 4 | 0.75653932 | 0.59985069 | 0.44846373 | |
| 5 | 0.76712029 | 0.61312912 | 0.47399084 | |
| 6 | 0.77477731 | 0.62136718 | 0.48668734 | 0.36743730 |
| 7 | 0.78045803 | 0.62673391 | 0.49410554 | 0.38204486 |
| 8 | 0.78474831 | 0.63035988 | 0.49893647 | 0.39073234 |
| 9 | 0.78803126 | 0.63287601 | 0.50222788 | 0.39482980 |
| 10 | 0.79056826 | 0.63466378 | 0.50453828 | 0.39924191 |
| $\infty$ | $\alpha = 0.8$ | $\alpha^2 = 0.64$ | $\alpha^3 = 0.512$ | $\alpha^4$=0.4097 |

Therefore this constant $\mu$ also satisfies the condition of local convergence of Theorem 2.1. Table 3 shows the convergent rates of the original sequence $\{x_n\}$ and the sequence $\{p_n^{(1)}\}$ accelerated by using the method for Limit Estimation. As shown in the Table 3, the rate $\frac{p_{n+1}^{(1)}}{p_n^{(1)}}$ converges to $\alpha^3 = 0.512$ instead of $\alpha^2 = 0.64$. This reason is that the function $\sin(x)$ is expanded to the power series as

$$\sin(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \cdots \quad (57)$$

at the root 0. In other words, there holds $a_{2k} = 0$ for $k \geq 1$ in (11). Therefore, from (25), we have $A_{2i} = 0$ for $i \geq 1$. Hence, there holds

$$x_n = \sum_{i=1}^{\infty} A_{2i-1}\alpha^{(2i-1)n} \quad (58)$$

from (23).

Table 3: Convergent rates of $x_n$ and $p_n^{(1)}$ when $\mu = 0.2$.

| $n$ | $x_n$ | $\frac{x_{n+1}}{x_n}$ | $p_n^{(1)}$ | $\frac{p_{n+1}^{(1)}}{p_n^{(1)}}$ |
|---|---|---|---|---|
| 0 | 1.00000000 | 0.83170580 | | |
| 1 | 0.83170580 | 0.82227333 | | |
| 2 | 0.68388950 | 0.81522962 | −0.38309828 | 0.48746614 |
| 3 | 0.55752698 | 0.81020137 | −0.18674744 | 0.49970617 |
| 4 | 0.45170912 | 0.80673232 | −0.09331885 | 0.50558146 |
| 5 | 0.36440834 | 0.80439715 | −0.04718028 | 0.50851516 |
| 6 | 0.29312903 | 0.80285187 | −0.02399189 | 0.51002994 |
| $\infty$ | 0 | $\alpha$=0.8 | 0 | $\alpha^3$=0.512 |

From these examples, we can confirm that the sequence $\{x_n\}$ of PCM is represented by the power series as follows:

$$x_n = p + \sum_{i=1}^{\infty} A_i \alpha^{in}. \quad (59)$$

## 6. Conclusion

PCM suffers from the slow convergence since the convergence is linear. Therefore the method itself is not useful. However, the method has a little calculation at every iteration since it dose not use a differential information. Furthermore, the sequence has a good property that the approximate sequence is represented by an infinite power series as $x_n = p + \sum_{i=1}^{\infty} A_i \alpha^{in}$ when the root $p$ of the nonlinear equation is not multiple. Here, $\alpha$ is represented by using a first differential coefficient $f'(p)$ as $\alpha = 1 - \mu f'(p)$. Based on this property, we got a better approximation which is estimated from the original sequence of PCM.

A problem left for the future is to analyze nonlinear equations with multiple roots. PCM can find an approximation of a multiple root of nonlinear equations. How does the sequence of PCM behave in the case of multiple roots? Another problem is extension of Theorem 3.2 to nonlinear equations of several variables. When we solve nonlinear equations of one variable, Newton-Raphson's method is one of the best choice to solve them. However, when we solve nonlinear equations of several variables by using the method, it takes much time to find a Jacobian matrix at every iteration. On the other hand, PCM uses an initial constant matrix instead of a Jacobian matrix and demands a little calculations at every iteration. Therefore we can hope that our approach of acceleration is extended to solving nonlinear equations of several variable.

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