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Quantum Chaos in Two-Level Systems Interacting with Resonant Radiation Field

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ABSTRACT

Quantum systems of Two-level atoms and radiation which corresponds to semiclassical systems where chaos occurs, are investigated, by using distribution functions obtained fron quantum master equations and its numerical analisis.
Key Word : QUANTUM CHAOS

1. Introduction

Chaos appearring in laser oscillation which was known from the early stage of laser research, was investigated theoretically by using the semiclassical approximation, on the basis of which, Maxwell-Bloch equation is usually used. [1] Fully quantum theoretical treatment might be preferable, for the theoretical completeness and the property of chaotic phenomena which is usually instability and sensitivity to the fluctuating disturbance that may be originated from quantum fluctuations. Quantum treatment of laser oscillation is, however, so complicated that adiabatic approximation to the atomic variable et.al. was used to reduce the dimension of equations, [2] which might be applicable for the chaotic or unstable oscilating systems, for which numerical research of multi-dimensional diffential equations has to be completed.

Another difficulty of quantum treatment of laser chaos are the existance of pumping and dissipation of energy of both atoms and radiation field. Former is especially needed in order to maintain laser oscilation suppling the energy transferred to radiation and escaped, but we have to give it pumping as the external system, which is not included lasing atoms and radiation field. That makes us impossible to describe the laser system in the form of closed dynamical systems defined by a definite hamiltonian. As a results, quantum hamitonian of laser system cannot be defined as a closed systems.

On the other hand, for atomic radiation systems resonantly interacting each other, it is seen that chaotic atomic motion appears when coupling between atoms and radiation is strong enough or a certain detuning frequency exists. [3] That was investigated numerically in the frame of semiclassical treatment, which is based on Maxwell-Bloch type equations, in which no pumpig is introduced. Fully quantum treatments are now tried for this closed systems which have definite hamiltonian of atomic and radiation field, without external force like a pumping et.al..

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In this paper, the review of semiclassical treatment of chaos in atomic resonant systems is given in the next chapter, and then quantum systems corresponding to these classical systems are investigated, according to c-number distribution functions obtained from the master equations for the quantum systems, by using numerical calculation of multidimensional equations.

2. Semi-classical chaos

Semi-classical treatment of resonant two-level systems, in which Electric field is described by classical variable, not the quantum operators, has been based on Maxwell-Bloch equations, as follows;

$$\dot{S}_x = -\omega_a S_y$$
$$\dot{S}_y = \omega_a S_x - 2\mu A S_z$$
$$\dot{S}_z = 2\mu A S_y$$
$$\ddot{A} + \omega_c^2 A = -2\mu\omega_c S_x$$

where S_x, S_y and S_z are the expectation value of spin operators, which correspond to the atomic two-level systems, which are described by the same equations as the spin systems. Corresponding spin systems may be seen like a top motion in the three-dimensional space, called bloch space, which we will use in quantum treatment. Resonant frequency of atomic levels ω_a, and one of electric field ω_c, appear explicitly without using rotating wave approximation, in which we couldn't

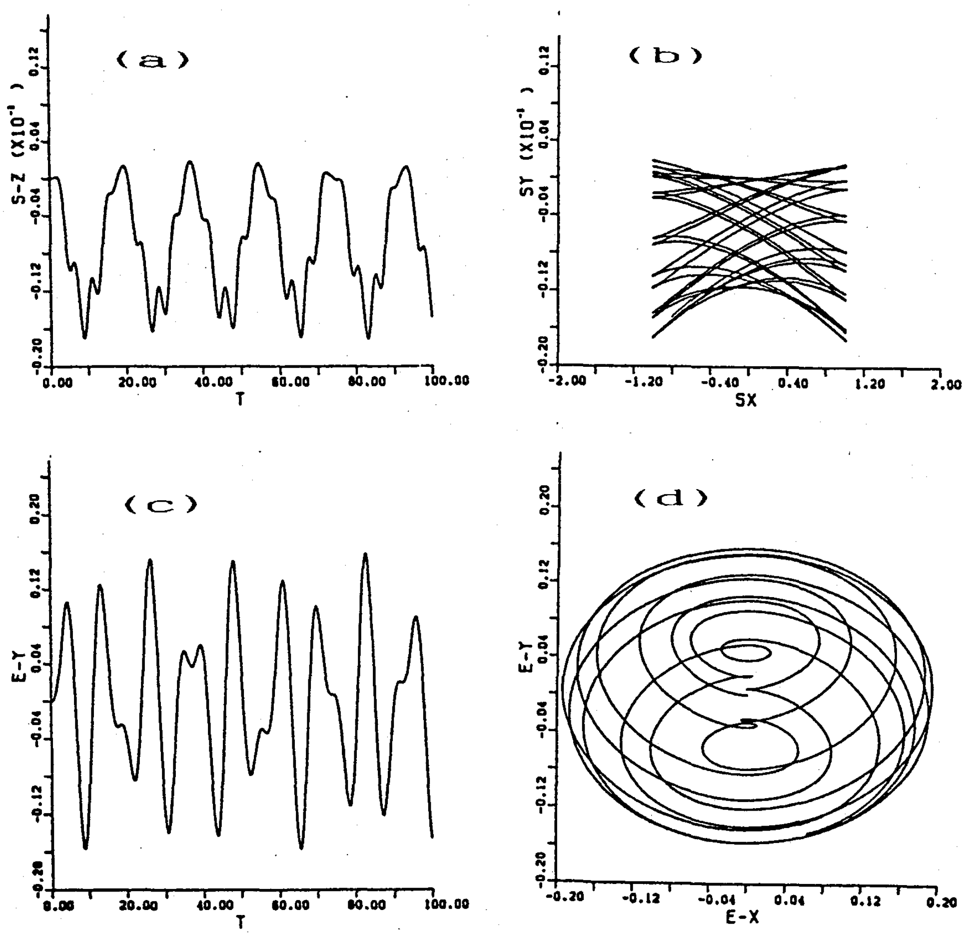

Fig.1 Variation and x-y trajectory of semiclassical spin variable and electric field. ωc=17711/28657, ωa=1, μ=0.03.

observe chaotic phenomena for the system of no pumpig. Couplig constant of elctric field and spins is now so-called bifurcation parameter for the chaotic motions, that is shown in result calculated later.

In the case of weak coupling, $\mu=0.03$, time variation of spin varaiable S_z, is periodic, that are shown in Fig.1 with the x-y trajectory of spin and field. For the case of $\mu=0.5$, spin behaves non-periodically, shown in Fig.2 with x-y trajectory, which shows typical chaotic motion. Motion of field may be seen stable in time variation, but chaotic in x-y trajectory. Another case of chaos occurence for the same system is the existance of frequency difference between atoms and field, where either of freqencys are not ineger, that may be connected to the origin of the chaos treated now, but not investigated sufficiently. Frequency spectrum or other parameter such as lyapunov characteristics et.al. are shown later, when they will be compared with quantum treatment.

3. Quantum theory

Now we construct quantum mechanichal systems corresponding to the former systems treated classically. Spin state is represented by the atomic coherent states, in which operators are described by polar coordinate of the Bloch space, which is similar to space of top motion.[4]

$$S^{+}\,|\,\Omega><\Omega\,|=e^{i\phi}[S\sin\theta+\cos^{2}(\theta/2)\,\partial/\partial\theta+i/2\cot(\theta/2)\,\partial/\partial\phi]\,|\,\Omega><\Omega\,| \quad (2)$$

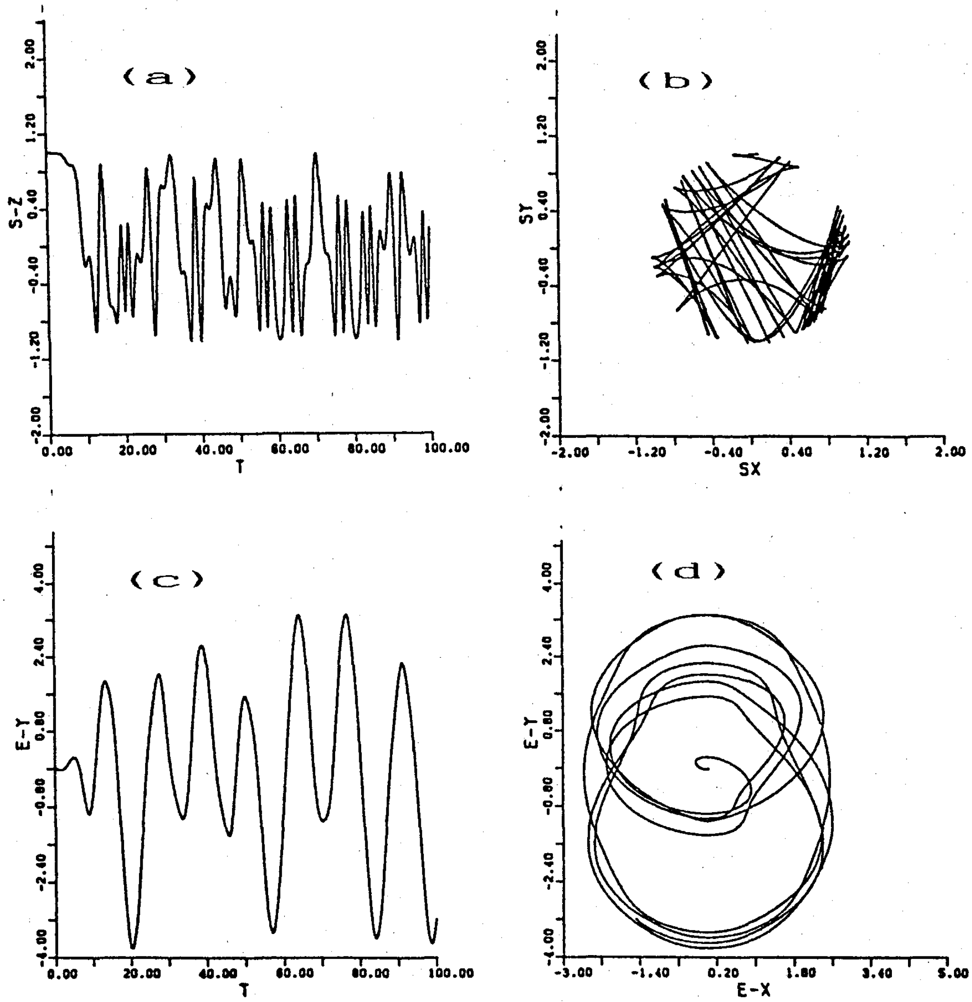

Fig.2 Variation and x-y trajectory of spin variables and electricfield. $\omega c=17711/28657$, $\omega a=1$. $\mu=0.5$.

Field state is now represented by coherent states, which corresponds to classical field amplitude E et.al., where the annihilation and creation operators are described by amplitude variabled as follow.

$$a^{+} \mid E\rangle\langle E \mid = (E^{*} + \partial / \partial E) \mid E\rangle\langle E \mid \qquad (3)$$

Interaction hamiltonian is expressed by the oprators and coupling constant μ, as follows;

$$H_{I} = ih\mu(a^{+}S - S^{+}a - Sa^{+} + aS^{+}) \qquad (4)$$

where we did not take the rotating approximation, by which the first two terms would have been left, that would make a drastic change to the chaotic behavior of the systems. Distribution function is used for describing the behavior of systems. Variable of that function are expectation value of quantum variable, that is, angular variables of Bloch space and field amplitude. That is obtained by expanding density operator, which is expanded by complete set of atomic coherent and field coherent states, whose coefficients are defined to be that distribution function.

$$\rho = \int d\Omega \int d^{2}E P^{c}(\theta, \phi, E) \mid \Omega\rangle\langle\Omega \mid \mid E\rangle\langle E \mid \qquad (5)$$

After tedious operator algebra we can obtain the partially differential equation for systems described quantum mechanically.

$$\begin{aligned}
\frac{\partial}{\partial t}P_c = & -\omega_c\left(\frac{\partial}{\partial E_y}E_x - \frac{\partial}{\partial E_x}E_y\right)P_c \\
& + 2\mu E_x\left\{\frac{\cos\phi\,\partial}{\sin\theta\,\partial\theta}(\sin\theta P_c) - \frac{\cos\theta\,\partial}{\sin\theta\,\partial\phi}(\sin\phi P_c)\right\} - 2\mu E_y\left\{\frac{\sin\phi\,\partial}{\sin\theta\,\partial\theta}(\sin\theta P_c) - \frac{\cos\theta\,\partial}{\sin\theta\,\partial\phi}(\cos\phi P_c)\right\} \\
& - \mu\frac{\partial}{\partial E_x}\left\{(S+1)\sin\theta e^{-i\phi} + \frac{1-\cos\theta}{2}e^{-i\phi}\frac{\partial}{\partial\theta} + i\frac{1-\cos\theta}{2}e^{-i\phi}\frac{\partial}{\partial\phi}\right\}P_c \\
& + \mu\frac{\partial}{\partial E_y}\{\text{ c. c. }\}P_c \qquad (6)
\end{aligned}$$

In order to solve this equation numerically we expand the unknown function by complete set of eigenfunctions, which are selected to be associate legendre functions for bloch space variable θ and, ϕ and hermite functions for the two dimensional space of electrical field variable, and as follows;

$$P_c = \sum_{l=0}^{\infty} \sum_{|m|\leq l} \sum_{n_x} \sum_{n_y} i^{m} a_{lmn_xn_y} Y_{l}^{(m)}(\theta, \phi) H_{n_x}(E_x) H_{n_y}(E_y) \qquad (7)$$

The expansion coefficients are now the unknown variables calculated numerically. It may be an advantage of those choise of eigen functions that the 1st order coefficients of legendre series with 0th order of hermit function and 1st order coefficients of hermite series with oth of regendre ones are equal to the classical expectation values of spin variables and field ones respectively, and further the associate legendre series terminate in a finite terms because the associate legendre functions is just the eigen functions of spin systems which is represented by bloch space.

As the above eq.(7)is linear partially diferential equation, the equations of unkwon coefficients have form of linear ordinary diferential equations with infinite unknowns, and constant coefficient, that might suggest that we could solve it by numerical linear transformation. But very large area of memory is needed for constructing the matrix, so we now solve those by Runge–Kutta formula of initial value problems. The above quantum system was, however, formally converted to a linear systems with infinite dimensions, where classical behavior was included as the lowest order partial space which cannot, however, be treated reconnected with the whole space.

4. Motions of classical variabes

Numerical values of lst order coefficients of either legendre series or hermite ones represent the motion of corresponding classicl variables. Results were obtained for the case of various coupling constant which make it possible to compare with the case of classical theory. Fig.3 shows the

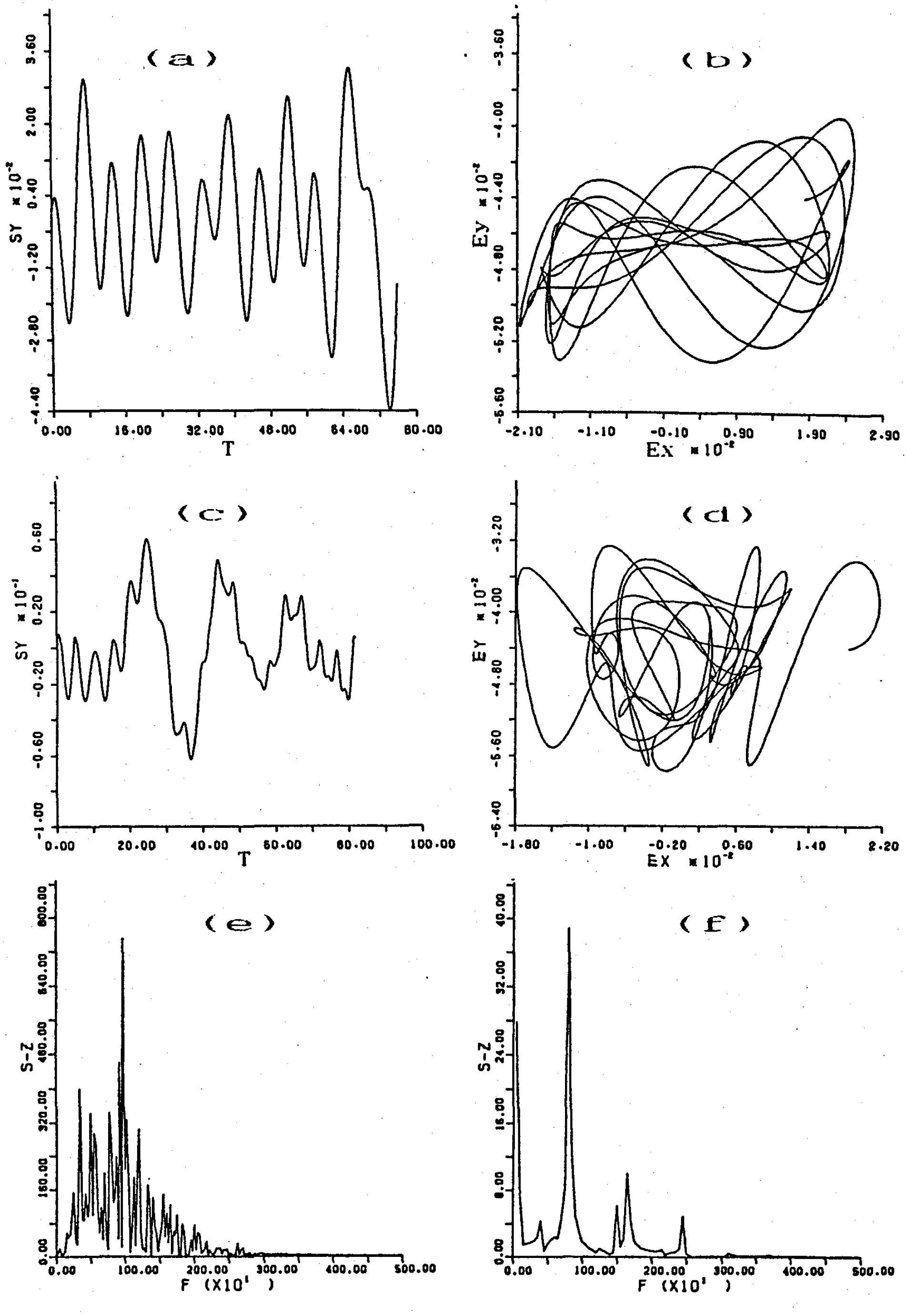

Fig.3 Variation, x–y trajectory and its fourier spectrum of spin variables and electric field. $\omega c = 17711/28657$, $\omega a = 1$. (a), (b) $\mu = 0.2$, (c), (d) $\mu = 0.5$, (e)spectrum of semiclassical variable $\mu = 0.5$, (f)spectrum of(c).

variation of expected spin variable and electrical field, and their x–y trajectory, for the same conditions as classical ones. Time variation may be seen to be high cut–offed filtered, that was also shown by their fourier spectrum which was simply obtained by fft programs for both semiclassical and quantum theoretical, shown in fig.3.

It has been observed in many models of chaotical systems that chaos is suppressed when classical systems were converted to corresponding quantum systems, that reveals in this case, where chaotic motion may not become extinct but may be hidden somewhere in the systems, where classical variable shown here, were included in only a partial space. In the case of stronger coupling constant, the same motion and its spectrum were represented, as the classical chaotic phenomena, as shown in fig.3.

Now we see the quantum paramenter which was included in higher terms than lst, but distribution function also shows the whole structure. Therefor, using the all coefficients, we calculate this function and show it by graphical technique, which were shown in fig.4, where the appearrence

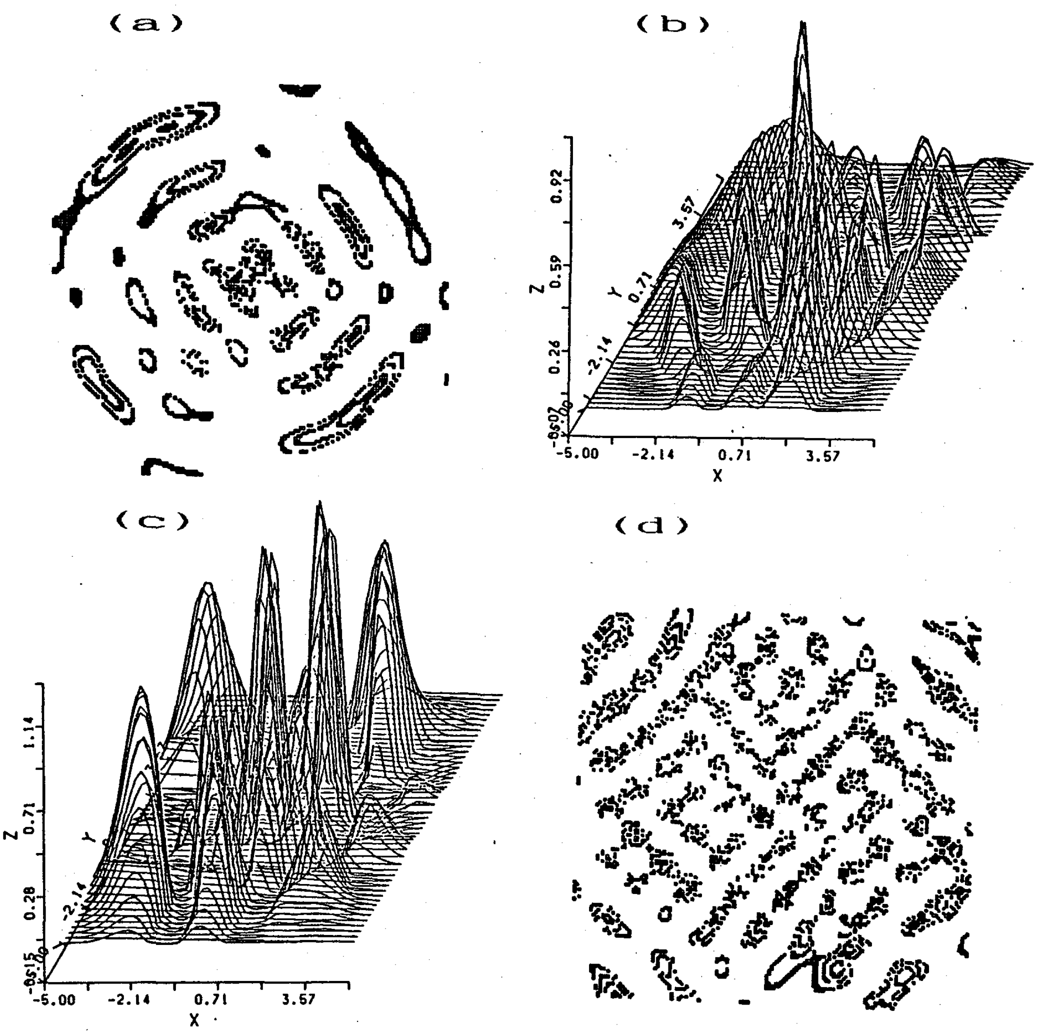

Fig.4 C–number distribution function and their contour lines of expected value of quantum electric field variable, at t=40, (a) $\mu=0.2$, (b) $\mu=0.5$(c), (d) $\mu=1.0$.

of function is seen to correspond to the classical case more than the lst order variables, that is, classical chaotic structure in the quantum system, appears in the shape of distribution functions, which is equivalent to wave functions through the conversion of represetation, not in the lst order parameter. Complexity of two dimensional function should be measuered quantatively.

In fig.4 the contour lines are also represented for the respecive distribution functions, and in fig.4 ones of the other case are also shown. The analogy with spin systems of fero-magnetism, suggest that the distribution will have self-similar or fractal patterns when the chaotic motion of system is occurring. Fractal dimensions obtained from the respective contour lines are nearly equal to an integer numbers and changed for all the cases of coupling constant, that shows that fractal dimension does not represent the complexity of distribution function, for which we must find the proper parameter to estimate the complexity of curved surface.

5. Motion of atomic functions

So for we have seen the statistical property of radiation field by using distribution function. We did not treat the atomic systems except time variation of its expectation value, because its distribution is trivial for the reason why we treat the case of single or a few atoms. The chaos of collective atomic two-level systems interacting with resonant field, might be ineresting as an example of simple quantum systems and the model of quantum systems of lasing atoms, such as superflourescence et.at..

We solve eq.(7) for the case of many atomic systems where spin valuable has eigen value more than one and the series of eigen function continues for many but finite terms. That makes the dimension of system too large to assign the valiables memory of computer. Now, using semiclassical approximation for the field valiable, eq.(7)is reduced to the follwing forms;

$$\begin{aligned}\frac{\partial P_c}{\partial t}=&2\mu E_x\left\{\frac{\cos\phi}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, P_c)-\frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\phi}(\sin\phi\, P_c)\right\}-2\mu E_y\left\{\frac{\sin\phi}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, P_c)+\frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\phi}(\cos\phi\, P_c)\right\}\\&+\frac{2\gamma}{\sin\theta}\frac{\partial}{\partial\theta}\left\{(S\sin^2\theta+1-\cos\theta)P_c\right\}+\frac{2\gamma}{\sin\theta}\frac{\partial^2}{\partial\theta^2}\left(\frac{1-\cos\theta}{2}\sin\theta\, P_c\right)\\&-\frac{2\gamma}{\sin\theta}\frac{\partial^2}{\partial\phi^2}\left(\frac{1}{2}\frac{\cos\theta\sin\theta}{1+\cos\theta}P_c\right)\qquad(8)\end{aligned}$$

where field variables obey semiclassical Maxell-Bloch equations whose atomic variables are defined to be expectation values.

Spin variable behave like a intermittent osclations whose spectrum may corresponds to the case of classical chaos. We have to notice that the coupling constant is 1.0 which is twice of the classical case. Field variable is seen not intermittently oscilate.

Distribution function of atomic variables fluctuate widely without reaching steady state. Its figure at a certain time are shown by contour lines, which was represented by small cells used for obtaining fractal dimensions by changing their sizes. Non-intger dimension was not obtained for all the case in fig.5. That may not result in a definite conclusion for the propertiy of distribution of atomic variablues, because of unsufficient precision of numerical calculation, which has to be improved in further research. Complexity of distribution is seen to depend on the coupling constant, which might be crrespond to chaotic property of the expection valuable, as the case of classical treatment.

Atom-atom correlation, which is partially given by the terms in eq.(8)and did not considered in this calcuations, that is, $\gamma=0$, might also be seen to give the property of chaos a drastic change, that is also left for a further calculations.

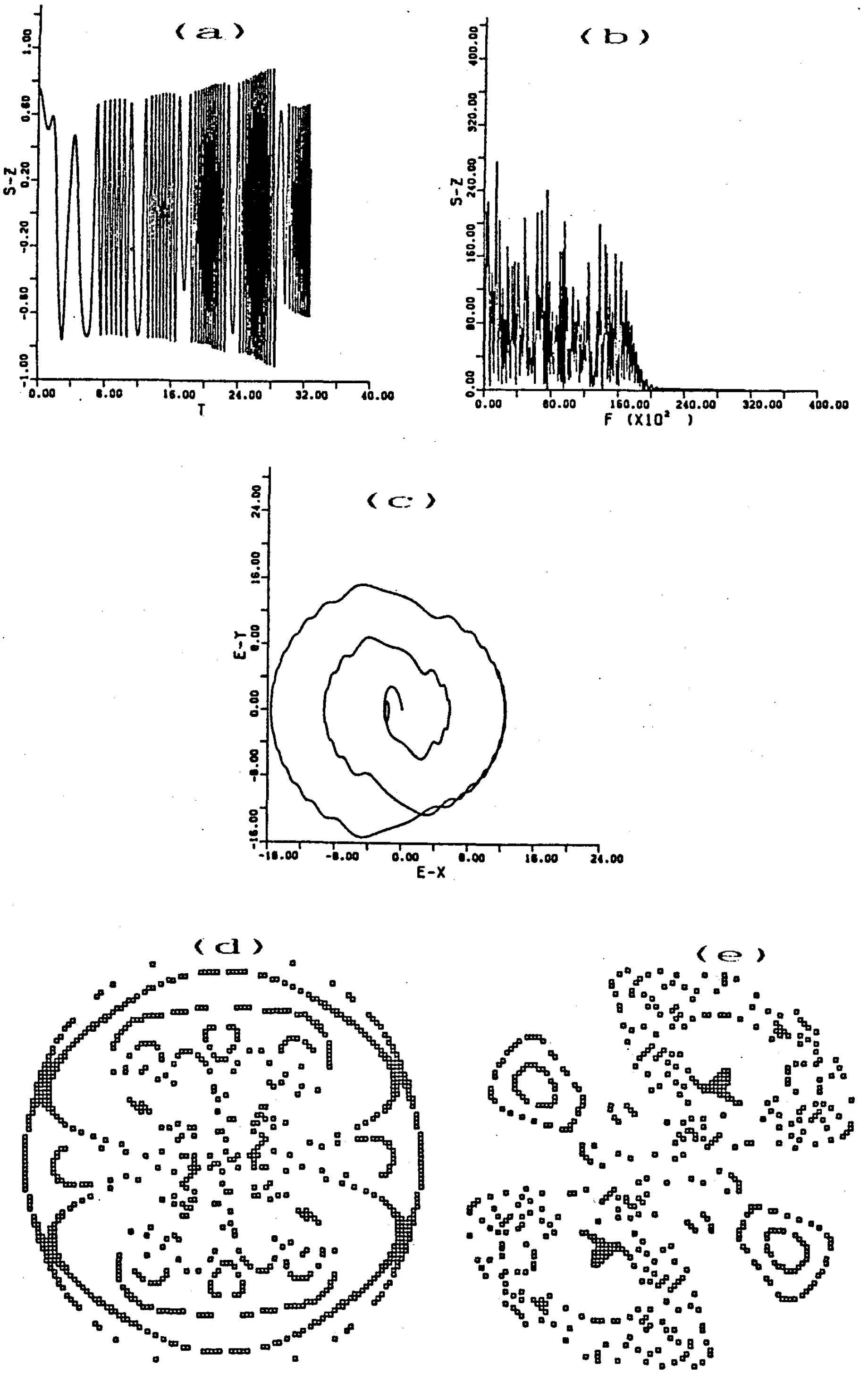

Fig.5 (a)variation of expected spin variable $\mu=1.0$(b)spectrum of (a), (c)x-y trajectrory of electric field, $\mu=1.0$, (d)contourline of distribution of spin variables, t=4, $\mu=0.5$, (e)t=8, $\mu=1.0$.

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