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Non-linear Behavior and Quantum Chaos in Laser and Cooperative Atomic Systems

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ABSTRACT

Chaotic behavior of laser oscillation is investigated by numerical calculation of quantum statistical equation of laser field, for the case of single mode and two-level systems. Relaxation-like distribution of laser field is expected from the quantum property of light. Further, coupled spin operators which are the base of atomic two-level system is investigated in expectation of chaotic behavior that might be origin of laser chaos.

Key Word: Quantum Chaos in Laser

1. Introduction

Chaos is a word assigned to phenomena whose behavior are non-deterministic but whose dynamical structure is deterministic. They were found, at early stage, in a solution of simple but nonlinear ordinary differential equations, which are approximated model of classical dynamical system, that is, hydrodynamics, known as Lorenz model[1]. Real phenomena of chaos, however, were observed from old times, for example, hydrodynamical turbulence and electrical oscillation *et al.* Until this time, a great variety of study of these chaotic phenomena were performed in various area of science, for example mathematical model called discrete dynamical system origined at Berunoulli map, classical oscillators, and hydrodynamics, for which author cannot refer all papers published.

We are now interested in optical chaos occurring at laser oscillation and non-linear interaction of coherent light and optical materials, one of which was observed at the early stage of laser as spiking phenomena of highly pumped laser, which was explained as a same type chaos as Lorenz system by Haken[2]. Chaos occurring in a delayed difference optical system is known as Ikeda chaos which was the first observed optical chaos[3].

Other optical chaos were observed and examined in many experimental conditions of laser oscillation and nonlinear optical systems[4]. These phenomena were rather classified as classical chaos because the equations are semiclassical coupled ones of mean values of quantum operators, which were transformed to polarization or population and electric magnetic fields. Chaos in quantum systems is the area of study which has a wide unknown parts. It is known that quantum system which corresponds to chaotic classical system, has a very

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complicated wave functions[5]. Coupled spin system also shows chaotic behavior in anti-ferromagnetism, that might be the most fundamental materials in which quantum chaos occurs[6].

In this paper semi-classical treatment of laser chaos and the comparison with quantum property of electrical magnetic field, coupled spin system which describe correlated atomic two-level systems and contribute laser oscillation, will be discussed.

2. Chaos in laser oscillation

The motion of electric magnetic field and atomic system which is usually described by two-level models, is treated by the following equations which is fully quantum mechanical[7];

$$\dot{a}_s = i\omega_s a_s - i\sum_i g_{is}^* (S_i^+ + S_i^-) \tag{1}$$

$$\dot{Q} = -i\omega_a [Q, \sum_i S_i^z] - i\sum_{j,s} g_{js} [Q, S_j^+ + S_j^-] a_s$$

$$- i\sum_{j,s} g_{js}^* a_s [Q, S_j^+ + S_j^-] \tag{2}$$

where all variable are Heisenberg operators, a_s is the anihilation operator of e.m. field of s th mode, where single mode will be assumed hereafter, s_i^+ is spin operators describing the atomic motion, one of which is written by Q. Now pumping effect must be added. That is given by flow-in of upper level atoms or transition from other levels. Fully quantum mechanical equations (1) and (2) do not include pumping effect which is introduced by averaging procedure such as that of the dispersion effect. As the result, c-number equations would be obtained by using tediuos algebra procedures, as follows[8];

$$\dot{E} = -(\frac{\gamma}{2} + i\omega_s)E + \mu\langle S^+\rangle \tag{3}$$

$$\langle \dot{S}^+\rangle = -(\Gamma_{12} + i\omega_a)\langle S^+\rangle + \mu\langle S^z\rangle E \tag{4}$$

$$\langle \dot{S}^z\rangle = R - \Gamma_{11}\langle S_z\rangle - \mu(E^*\langle S^+\rangle - E\langle S^+\rangle^*) \tag{5}$$

where all variable mean the c-numbers instead of operators. R is the rate of pumping, that is, the rate of increasing of population difference between upper atomic level and lower one which cannot be described only by pure spin operator systems. $\langle S\rangle$, the expectation value of spin operator, is generally complex number, population difference corresponds to S^z of spin operators. Relaxation coefficients γ, Γ have the usual meaning of laser physics, that are given by averaging procedure, too.

Those equations are type of non-linear coupled ones, which are solved by numerical method of the initial value problems. **Fig. 1** shows the variation of electric field intensity, that shows random oscillation. Such random behavior was observed experimentally in highly pumped laser which gives very high intensity, such as ruby laser or neodium glass *et al.* At the early stage the control of these instability, and stable oscillation was prefered, that is,

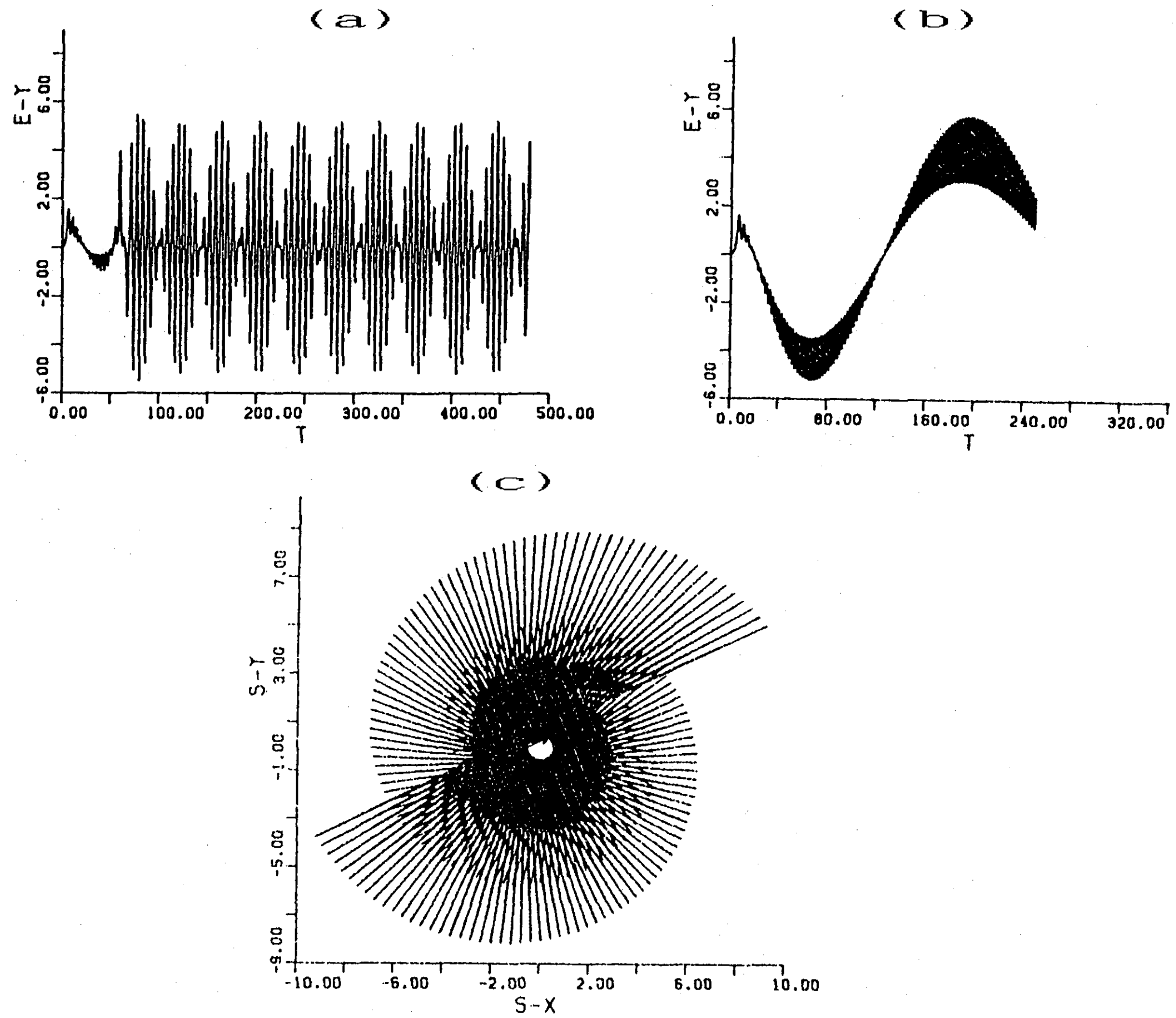

Fig. 1 Laser field trajectry for eqs. (3)–(5) $\Gamma_{12}=\Gamma_{11}=0.01$, $\Upsilon=0.1$, R=5, $\mu=0.5$, (a) $\gamma=0.1$, (b) $\gamma=0.05$, (c) x–y plot $\gamma=0.05$.

Q-switch was invented. But the details of the mechanism of these random behavior was neglected.

As for the laser oscillation, these randomness or so-called chaos is observed for the intensity of field but not for the phase of laser oscillation, that is one of the characteristic of laser chaos, as shown in fig. 1, where trace of laser electric field is shown by phase diagram, which rotates with a constant phase according to the frequency difference between the cavity and the atomic level.

The conditions for chaotic behavior of laser oscillation can be obtained by the stability analysis of linear systems, which is driven by linearlization of original ones, that is known as bad cavity conditions[9], but give only threshold of chaos, not the quality of occurred chaos. That can be investigated only by numerical methods. Random spiking phenomena of laser intensity might be obtained wrong by an unexact difference method which gives unstable or chaotic solution to the stable systems. Stability of numerical algorism has to be checked, but stability of both systems are connected and cannot be treated separately.

As Eq. (3), (4) and (5) are of type of semiclassical non-linear equations, the chaos given now was classified to classical one. As we are interested in quantum chaos in laser

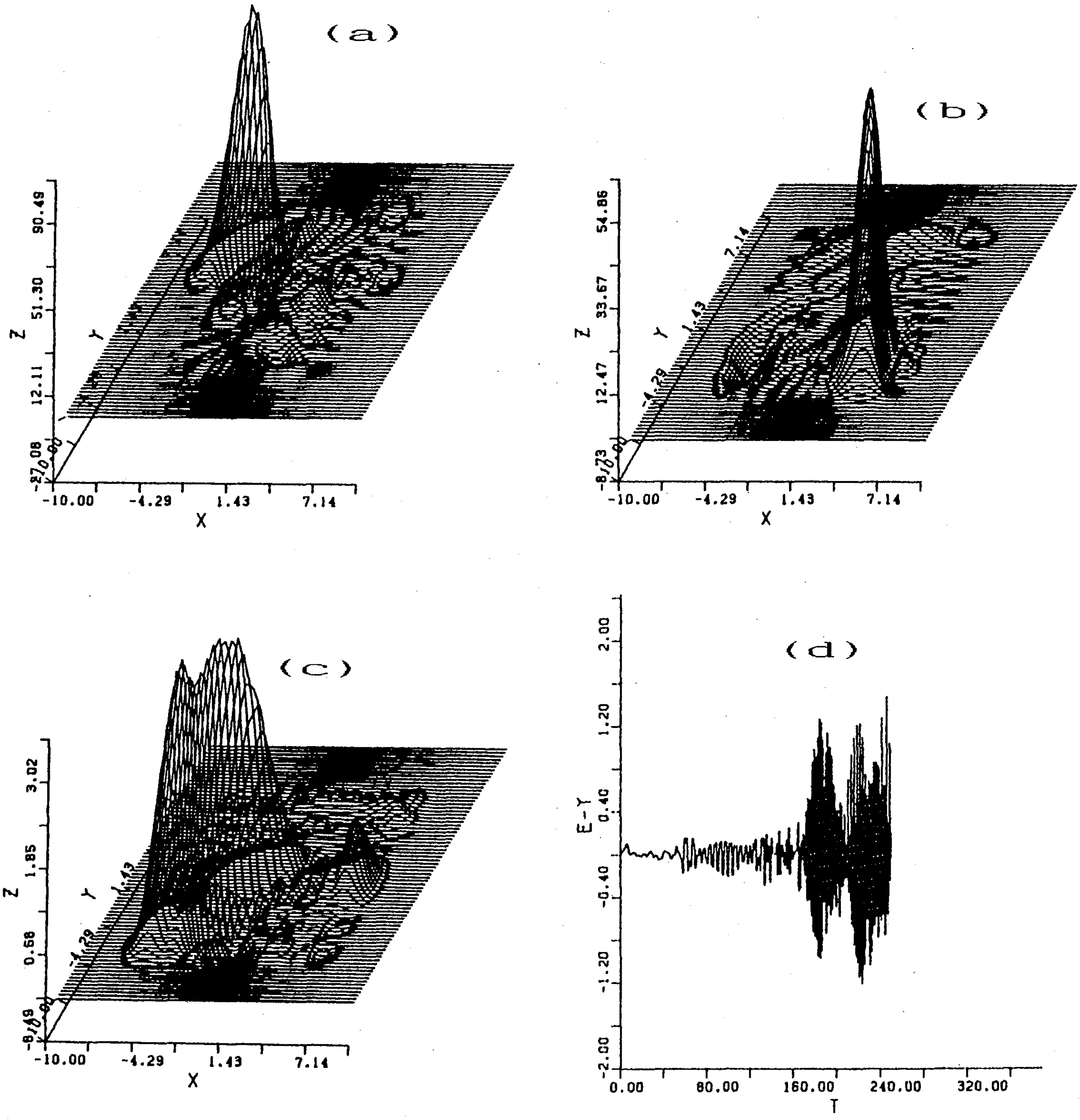

Fig. 2 Time evolution of eq. (6), for same parameter as fig. 1 (b), (a) t=15, (b) t=30, (c) t=45, (d) x–y plot of field by eq. (6).

oscillations, fully quantum mechanical equations have to be solved. Quantum treatment of laser theory was formulated by using density matrix of photon state, or by using master equations which lead to distribution of expectation value of quantum operators. Now we consider chaos according to these master equations. After tedious operator algebra, which is not written here, eq. (1) and (2) are transformed to c-number representative, as follows[8];

$$\frac{\partial P_c}{\partial t} = -\frac{\partial}{\partial E}\,[\,\mu\langle S^{+}\rangle - (\frac{\gamma}{z} + iw_s)E\,]\,P_c + c.C. + \gamma\frac{\partial^2}{\partial E\partial E^*}P_c \qquad (6)$$

where E and *et. al.* represent mean value of each operator, instead of operators. As for laser electric field E, the equation for distribution function is the same as the semi-classical ones, which shows that quantum effect does not appear in electric and magnetic field by themselves, but through atomic system which is perfectly described by quantum mechanics. Interested in the behavior of electric magnetic field, taking the mean value of atomic variables, we obtain equations of distribution function for only e.m. field, which couples to the mean value of atomic variables.

Two-dimensional distribution function can be solved by numerical method. Expanding the distribution function by the complete set of hermite polynomials, the problem is transformed to equations of expansion coefficient of each polynomial, that will be solved by usual numerical method of ordinary differential equations. Especially coefficients of 1-st hermite polynomial are the mean value of field E, that can be compared with the behavior of semi-classical systems. Property of distribution function is seen from the 3-dim. graph obtained by the calculated expansion coefficients of hermite polynomials. Variation of distribution is similar to the relaxation of energy distribution of molecules. Gauss function might be a exact final form, which is 0-th hermite eigenfunction. These behavior can be explained by the diffusion like term of eq. (6), which makes the coefficients decay except the 0-th terms, that suggest that the system of expansion coefficient might be stable, but corresponding system of laser oscillation is chaotic. Numerical error cannot be neglected for the above 3-dim. calculation, for example, when 20x20 terms of hermite polynomials were used, negative value of distribution function occurs, locally, therefore 3-dim. graph of distribution may be qualitative. Quantitative treatment such as correlation function, fractal dimension *et. al.* will be in future work.

3. Collective atomic systems

It may be seen that chaotic property of laser oscillation does not come from laser radiation, but from atomic transition which contributes to the laser radiation. So, quantum property of laser chaos will be seen by considering the atomic motion where chaotic motion is occurring. These atomic motion are usually collective as seen from the phenomena such as superfluorescence *et al.* These collective motion is described by coupled spin operators, when two-level and collective system is assumed. To study collective resonance fluorescence, these spin operators were used, where they were described by bloch equations, but also treated by distribution function of mean value of operators[10]. On the contrast to the case of laser equation, these collective spin operator was treated fully quantum mechanically without averaging procedure such as the dispertion *et al.* The equation of distribution function of mean values of spin operator S is given by operator algebra which is not written here, as follows [11];

$$\frac{\partial P}{\partial t} = -\frac{2\mu}{\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\varphi\sin\theta)P) + \frac{\partial}{\partial\varphi}(\cos\theta\cos\varphi P)\right]$$
$$+\frac{2\gamma}{\sin\theta}\frac{\partial}{\partial\theta}\left[\left(\frac{N}{2}\sin^2\theta + 1 - \cos\theta\right)P\right] + \frac{2\gamma}{\sin\theta}\frac{\partial^2}{\partial\theta^2}\left(\frac{1-\cos\theta}{2}\sin\theta P\right)$$
$$-\frac{2\gamma}{\sin\theta}\frac{\partial^2}{\partial\varphi^2}\left(\frac{1}{2}\frac{\cos\theta\sin\theta}{1+\cos\theta}P\right) \qquad (7)$$

About the mean value <S> *et al.*, the equations which was known as coupled Bloch equations, will be reduced as follows [12];

$$\langle\dot{S}_i^{\pm}\rangle = [\pm i(w_a - w_s) - \gamma]\langle S_i^{\pm}\rangle + 2\sum_{k\neq i}\gamma_{ik}\langle S_k^{\pm}\rangle\langle S_i^{z}\rangle$$
$$\pm 2i\mu\langle S_i^{z}\rangle \qquad (8)$$

$$\langle\dot{S}_i^{z}\rangle = -\sum_{k\neq i}\gamma_{ik}(\langle S_k^{-}\rangle\langle S_i^{+}\rangle + \langle S_k^{+}\rangle\langle S_i^{-}\rangle)$$
$$+ i\mu\langle S_i^{+}\rangle - i\mu\langle S_i^{-}\rangle - 2\gamma\langle S_i^{z}\rangle \qquad (9)$$

where so-called mean field approximation was usually assumed, that is <S S>=<S> <S>, which means that the interaction between different atoms is classical. Of course, exact quantum mechanical treatment would be possible, if the interaction between atoms is homogeneous, that leads to extended Bloch equations whose spin operators have more value than two, which are similar to angular momentum of atomic electrons.

General atom-atom interaction is, however, not homogeneous, and resonant frequency of atomic transition may not be equal each other. At this case, the mean field approximation is usually used. The distribution function of mean field approximation is non-linear and self-consistent.

To see temporal behavior of correlated two-level systems, coupled Bloch equations (7), (8) and (9) are solved numerically. The solution are shown graphically by using trace diagrams of phase space of S_x and S_y which are real part and imaginary part of mean value of spin operator respectively. At the case of homogeneous coupling and no detuning distribution of resonant frequency, the trace in phase space shows periodic motion as shown in **fig. 3**, therefore the temporal behavior is also periodic. At the case of coupling of nearest neighbors, however, the trace shows drastically different patterns which might be seen to be non-periodic and unstable orbit, which is shown in fig. 3. Non-periodic behavior of spin operator is seen at the case of different resonant frequency of each coupled atoms. We cannot conclude that these behavior of spin are chaotic, but the analogy to classical coupled oscillator, which have chaotic or multi-periodic motion, suggest the chaotic quantum motions. As the distribution function of spin operator, eq. (7) or one modified to self-consistent ones, must be solved numerically that needs 2-dim. analysis. These distribution function can be expanded with a series of legendre bi-functions, which term in a finete terms, that suggests

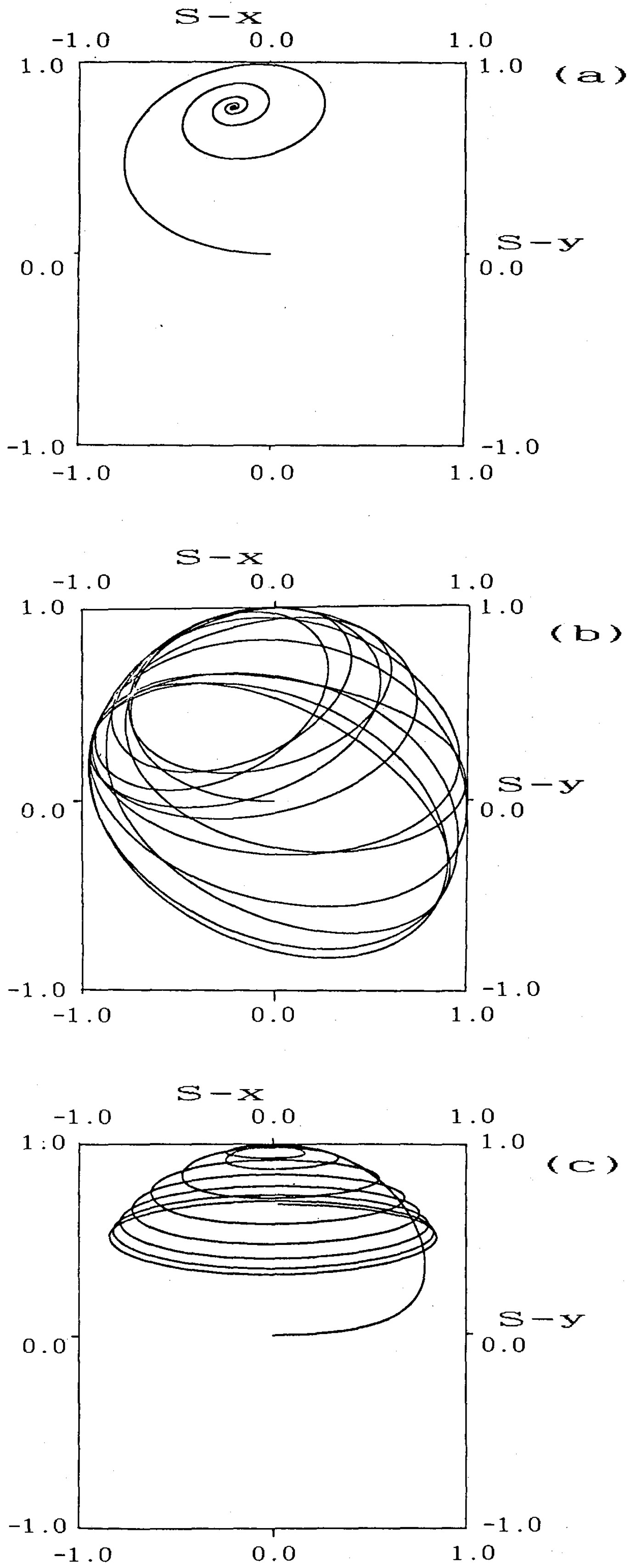

Fig. 3 x-y plot of spin operator (a) Stable orbit of three homogeneous coupled spins, $\gamma_{ij}=0.1$, $\mu=1.0$, $t=32$, (b) Unstable orbit of nearest neighbor coupled, $\gamma_{ij}=0.5$, $\mu=1.0$, (c) Orbit of different detuning (0.1×10) $\gamma_{ij}=1.0$.

chaotic patterns in phase space would not occur. Final conclusion would be given after more detailed numerical calculation and mathematical consideration.

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