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General Equilibrium Model of Optimal Transportation Supply

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ABSTRACT

This paper is concerned with general equilibrium model of optimal supply of transportation facilities and optimal pricing to transport services. Strotz's work (1965) that studied optimal road supply and congestion toll on a single one-way road is developed to the more general case including competitive transport services. Here we consider two competitive roads, say free general road and toll-charged expressway, and public transport. Both optimum allocation of resources and optimal pricing to these three routes are discussed by use of general-equilibrium-theoretic approach.

The conceptual framework used in the work is mostly the same as in the Strotz's model, but partially different. Major improvements are as follows: (1) time budget is introduced into the set of constraints of household utility-maximizing behaviour, (2) consumers (=trip makers) are faced to three competitive routes, so trip costs that were ignored in the Strotz's model are evaluated and fuel tax is also introduced into the model as a new financial source for transportation investments.

Main results involve the so-called marginal cost pricing principle, e.g. the expressway's toll must be equated to the balance of the two marginal rate of substitution between trips and resource input to two roads. As for the public transport fare this principle also holds. Finally, a process for approaching general equilibrium solutions is illustrated.

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1. INTRODUCTION

This paper is concerned with general equilibrium model of optimal supply of transportation facilities and optimal pricing to transport services. Strotz's work (1965) that studied optimal road supply and congestion toll on a single one-way road is developed to the more general case including alternative transport services. Here we consider two competitive roads, say free general road and toll-charged expressway, and public transport. Both optimum allocation of resources and optimal pricing to these three routes are discussed by use of general-equilibrium-theoretic approach.

This paper deals with optimum allocation of resources in transportation, but it does not deal with the distributional effects. Nor are we here concerned with methods for evaluating specific proposals by measuring benefits and costs. Moreover, our concern is with intra-urban rather than interurban transportation.

A conceptual framework used in the paper is mostly the same as in the Strotz's model, but partially different. An attempt to develop and improve his model, which is strongly colered by the standpoint of welfare economist, was made especially in the following manners: (1) travel demand is treated as a derived demand from activities, so our concern is focused upon journeys to work, school and shop, (2) time budget is introduced into the set of constraints of household utility-maximizing behaviour, (3) both consumption of private goods and leisure time raise the level of utility, (4) consumers (trip makers) are faced to the three routes mentioned above, (5) trip costs that were ignored in the Strotz model are evaluated, and fuel tax is also introduced into the model as a new financial source for transportation investments.

Various methods of employing travel budgets to improve conventional travel demand models have been explored (e.g. Beckmann, Golob (1974); Golob, Beckmann, Zahavi (1981), etc.). These have actually made contributions to the theory of travel demand like deriving gravity model from utility theory. The purpose of the research reported in this paper is to approach the problem of optimal transportation investment in the frame of general equilibrium model proposed by Strotz, with linkages to the modal choice and travel budgets in a manner consistent with behavioural science.

2. THE MODEL

Now to start formulating the idealized problem. Suppose there are three routes, say free general road, toll-charged expressway and public transport. Each leads away from a population (concentrated at a point), turns about, and comes back again, perhaps a round-trip route from a center of residential area to the CBD. Therefore, the term "trip" used in the paper means a home-based round-trip to and from a destination.

We assume the economy which comprise household sector, private goods production sector and public sector (government). The manner of behaviour of these sectors are formulated as follows.

2.1 Household Sector

The i-th household sells its resource (labor) of amount R_i in the market to get income under the constant wage rate w. Disposal income I_i is then defined as

$$I_i = w R_i - H_i \qquad (i = 1, 2, \ldots, N) \qquad (1)$$

where H_i is income tax depending on w R_i.

Household is also assumed to have a utility function

$$u^i = u^i (x_i, L_i) \qquad (2)$$

where u^i is a utility indicator, x_i is the amount of all kind of private goods except trips (we call it "composite commodity"), and L_i is the leisure time. The partial derivative of this function with respect to x_i, $\partial u^i/\partial x_i$, to be indicated as u^i_x, is positive in the relevant range, and $\partial u^i/\partial L_i$, indicated as U^i_L, is also positive[1]. Thus, the more consumptions and leisure time which increases with time savings gained by the faster travel-mode use, the higher the utility level.

Now given the travel demand Q_i for household i, following equation holds:

$$Q_i = t_i + T_i + P_i \qquad (3)$$

where t_i, T_i and P_i are the trip times via free road, toll road and public transport, respectively. The pattern of trip mentioned above is illustrated in Fig. 1.

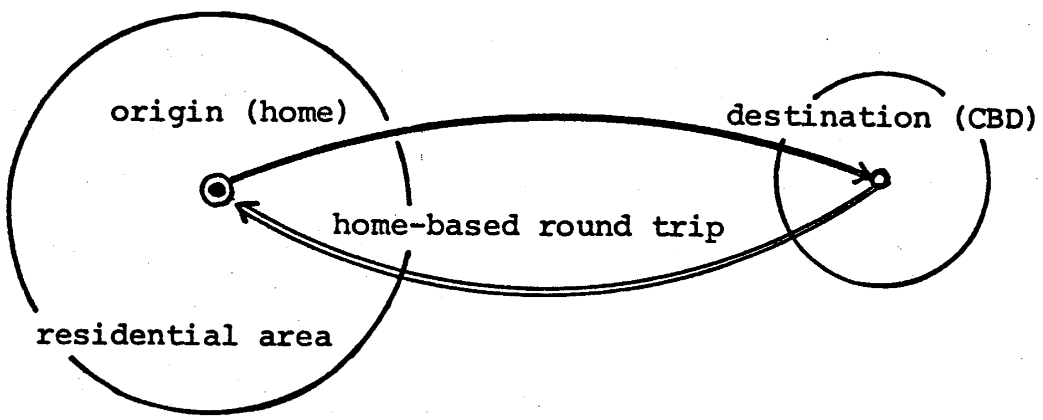

Fig. 1 : Pattern of Travel Demand

Household is assumed to have two constraints in maximizing its utility indicator. One is the budget constraint

$$I_i = \pi x_i + \phi t_i + (\phi + \tau) T_i + \psi P_i \qquad (4)$$

where π is the price for unit composite commodity, ϕ is the fuel cost per trip in auto use, ψ is the transit fare per trip, and τ is the expressway toll per trip. For simplicity it is assumed that there is no difference in fuel cost between free road and toll road.

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1) Throughout this paper we assume that the usual convexity conditions for a maximum hold.

Another is the time budget constraint

$$K_i = L_i + q\,t_i + r\,T_i + s\,P_i \tag{5}$$

where q, r and s are the travel time via free road, toll road and transit, respectively, and K_i is the given available time for leisure and travel, which is the balance that the total clock time for the analysis period multiplied by the number of persons in the household minus time required for nondiscretionary activities (working, sleeping, eating, etc.). Possible choice set of travel modes is shown in Fig. 2.

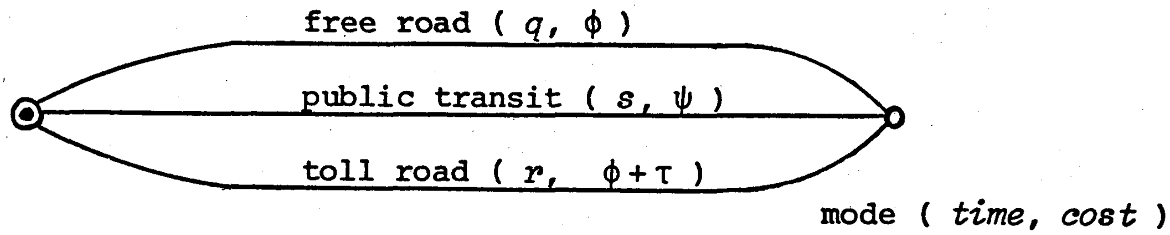

Fig. 2 : Choice Set of Travel Modes

There is one specification as Strotz did and we shall consider in more detail later: that is whether or not q and r, though the same for all individuals, is regarded by each as a parameter, that is, as a given value which he cannot alter. We shall in any case have q and r depend upon the total number of trips made per unit time by all individuals in the community, $\Sigma\, t_i$ and $\Sigma\, T_i$, but we might wish to suppose that each individual assumes that q and r do not depend on his own frequency of travel.

Thus, household utility-maximizing problem may be formulated as

$$[\text{ Maximize (2) subject to (3), (4) and (5)}]. \tag{6}$$

However, we can solve t_i and L_i as a function of variables T_i and P_i from Eqns. (3) and (5).

$$t_i = Q_i - T_i - P_i$$
$$L_i = K_i - q\,Q_i - (q - r)\,T_i + (q - s)\,P_i$$

Then, utility function (2) is to take a new form of

$$\begin{aligned} u^i &= u^i(x_i, L_i) \\ &= u^i(x_i, K_i - q\,Q_i - (q-r)\,T_i + (q-s)\,P_i) \\ &= \hat{u}^i(x_i, T_i, P_i) \end{aligned} \tag{7}$$

and constraint equation (4) also becomes

$$I_i = \pi\, x_i + \tau\, T_i + (\psi - \phi)\, P_i + \phi\, Q_i. \tag{8}$$

Then, our problem is simply expressed as

$$[\text{ Maximize (7) subject to (8)}]. \tag{9}$$

This problem can be solved by the method of Lagrangean multipliers. Consequently, the first order conditions for maximum, that is subjective equilibrium condition, are summarized

$$\hat{u}^i_T/\hat{u}^i_X = \tau/\pi \quad \text{and} \quad \hat{u}^i_P/\hat{u}^i_X = (\psi-\phi)/\pi. \tag{10}$$

Equation (10) means that the individual, maximizing utility subject to a fixed unit toll and fixed unit price of composite commodity, therefore equates his marginal rate of substitution between trips and composite commodity to the price ratio.

2.2. Commodity Production Sector

The model of the composite commodity production sector is entirely same to Strotz's. Its market clearing condition is

$$\Sigma x_i = f(E_0) \tag{11}$$

where f is a production function and E_0 is the amount of resource input to produce composite commodity. The derivative of this function, f′, is the marginal productivity of resource services in producing composite commodity; it is assumed to be positive, and is assumed, in the relevant range, not to increase with an increase in E_0. Being the price of unit resource w, the profit of the firm may be written as

$$\Pi = \pi\,\Sigma x_i - w\,E_0 = \pi\,f(E_0) - w\,E_0.$$

Thus, the first order condition for profit-maximization is $d\Pi/dE_0=0$, and then

$$f' = w/\pi, \tag{12}$$

that is an efficient condition for production.

On the other hand, as for the fuel production sector, a similar formulation as above can be done. Its market clearing condition is

$$\Sigma\,(t_i+T_i) = y(E_1) \tag{13}$$

where y is a production function, E_1 is the amount of resource input and $\Sigma(t_i+T_i)$ is the fuel demand. As a profit of the firm is expressed as

$$\begin{aligned}\Pi &= (1-\mu)\phi\,\Sigma(t_i+T_i) - wE_1\\ &= (1-\mu)\phi\,y(E_1) - wE_1,\end{aligned}$$

then from the first order condition for profit maximization, $d\Pi/dE_1=0$, we have

$$y' = w/[(1-\mu)\phi]. \tag{14}$$

where μ is a fuel tax rate.

2.3. Public Sector

Annual revenue in public sector (government) consists of income tax from households, fuel tax from auto users, toll revenue from expressway users and fare revenue from transit users. Under this financial source, the government purchases the resource (labor) only to supply transportation services.

An objective function which society wants to maximize is assumed to be a social welfare function of the general form

$$w = w(u^1, u^2, \ldots, u^i, \ldots, u^N) \qquad (15)$$

where u^i is the level of the utility index of the i-th household.[2] The goal of society is to maximize (15) subject to the constraints given by technological relationships including production functions. There is another independent technological relationship, which we shall call the congestion function or travel time function

$$q = q(\Sigma t_i, E_2) \qquad (16)$$

specifying the driving time per trip as a function of the total number of trips per unit time and the annual expenditure E_2 to the general road. The partial derivative of this function with respect to the total volume of traffic, $\partial q / \partial \Sigma t_i$, to be indicated as $q_{\Sigma t}$, is positive in the relevant range, and $\partial q / \partial E_2$, indicated as q_{E_2}, is negative.

Thus, for a given road expenditure, the more trips, the greater the congestion; for a given number of trips, the greater the road expenditure, the less the congestion.[3] Similarly for the expressway

$$r = r(\Sigma T_i, E_3), \quad r_{\Sigma T} > 0, \quad r_{E_3} < 0 \qquad (17)$$

and for public transport

$$s = s(\Sigma P_i, E_4), \quad s_{\Sigma P} > 0, \quad s_{E_4} < 0. \qquad (18)$$

On the other hand, the total amount of resources which exists in the society is $R = \Sigma R_i$. Then the market clearing condition for resource becomes

$$E_0 + E_1 + E_2 + E_3 + E_4 = R. \qquad (19)$$

Using Eqn. (19), market clearing condition for composite commodity is changed

$$\Sigma x_i = f(R - E_1 - E_2 - E_3 - E_4). \qquad (20)$$

Thus, the problem of the society can be formulated as

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2) The form for a social welfare function is not unique. It depends upon the value judgements of its formulators. It might be derived from a common consensus.

3) The relation between driving time and traffic volume may be illustrated as in Fig. A. Usually, the greater part of the sparse dotted line cannot be observed. In a work of route assignment, linear travel time function is often used instead of the real line of the curve. That is

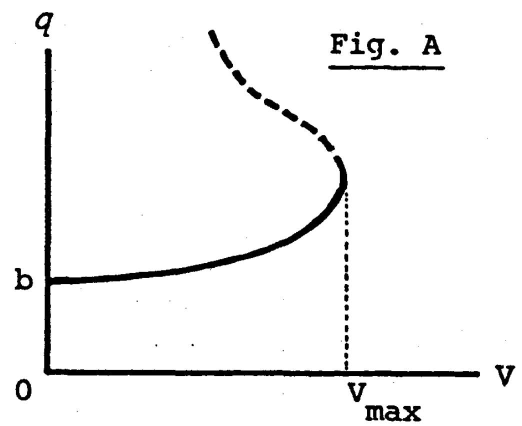
$$q = aV + b$$

where q is a travel time through the link, V is a traffic volume per time, a is a parameter that represents width of road, and b is a travel time at V=0. Then, the use of the form

$$q = \hat{a} \, \Sigma t_i / E_2 + b$$

may be convenient, where $\hat{a}/E_2$ corresponds to parameter a. This is a function homogeneous of degree zero.

Fig. A



$$[\text{Max. } (15) \text{ sub. to } (13), (16), (17), (18) \text{ and } (20)]. \tag{21}$$

The problem may be solved also by the method of Lagrangean multipliers. Consequently, the first-order condition for maximum is

$$\hat{u}^i_T / \hat{u}^i_X = f' \cdot (q_{\Sigma t}/q_{E_2} - r_{\Sigma T}/r_{E_3}). \tag{22}$$

Substituting Eqns. (10) and (12) into Eqn. (22), we have

$$w \cdot (q_{\Sigma t}/q_{E_2} - r_{\Sigma T}/r_{E_3}) = \tau. \tag{23}$$

The implication of Eqn. (23) is quite important. The expressway's toll must be equal to the balance of the two marginal rates of substitution between trips and resource input. In another words, the left-hand side of Eqn. (23) is the balance between the additional resource input required for cancelling the increase of travel time which was induced by the unit trip increase in the general road and that of the expressway. Thus, Eqn. (23) means that the expressway's toll must be equal to such a balance as above, that is so-called marginal cost pricing principle.

Moreover, we can derive another efficient condition as

$$\hat{u}^i_P / \hat{u}^i_X = f' \cdot (q_{\Sigma t}/q_{E_2} - s_{\Sigma P}/s_{E_4} - 1/y'). \tag{24}$$

In the same manner as in Eqn. (22), substituting Eqns. (10), (12) and (14) into Eqn. (24), we have

$$w \cdot (q_{\Sigma t}/q_{E_2} - s_{\Sigma P}/s_{E_4}) = \psi - \phi\mu. \tag{25}$$

The implication of Eqn. (25) is also important. The left-hand side of Eqn. (25) means the balance of additional resource input required for cancelling the increase of travel time which was induced by the unit trip increase both in the general road and in the public transport. The right-hand side of Eqn. (25) is the balance between realized fare revenue and unrealized fuel tax per trip. In another words, Eqn. (25) means that marginal cost must be equal to marginal revenue.

2.4. Financial Balance in Public Sector

Annual revenue of the public sector, indicated as AR, is the sum of the revenues of income tax, fuel tax, expressway's toll and transit fare. That is

$$AR = \Sigma H_i + \phi\mu \cdot \Sigma (t_i + T_i) + \tau \cdot \Sigma T_i + \psi \cdot \Sigma P_i. \tag{26}$$

On the other hand, annual expenditure, indicated as AE, is

$$AE = w \cdot (E_2 + E_3 + E_4). \tag{27}$$

From Eqns. (1) and (4), we have

$$w R_i - H_i = \pi x_i + \phi t_i + (\phi + \tau) T_i + \psi P_i. \tag{28}$$

Summing Eqn. (28) for all i,

$$w \Sigma R_i = \Sigma H_i + \pi \Sigma x_i + \phi \Sigma t_i + (\phi + \tau) \Sigma T_i + \psi \Sigma P_i. \tag{29}$$

From Eqn. (19), multiplied by w, we have

$$w\cdot(E_2 + E_3 + E_4) = w\,\Sigma R_i - w\,E_0 - w\,E_1. \qquad (30)$$

Substituting Eqn. (29) into (30),

$$\begin{aligned} w\cdot(E_2 + E_3 + E_4) = & [\,\Sigma H_i + \tau\,\Sigma T_i + \mu\phi\Sigma(t_i+T_i) + \psi\,\Sigma P_i] \\ & + [\,\pi\Sigma x_i - wE_0\,] \\ & + [\,(1-\mu)\cdot\phi\Sigma(t_i+T_i) - wE_1\,]. \qquad (31) \end{aligned}$$

The second brace of the right-hand side of Eqn. (31) is the excess profit of the composite commodity production sector, and the third the excess profit of the fuel production sector. Both of them must be equal to zero under the perfect competition market equilibrium. Therefore, Eqn. (31) means that AE=AR, that is a balanced budget. Needless to say, this is nothing but *Walras' law* that, in an economy of m markets, if equilibrium is attained in (m–1) markets, it is automatically attained in the m–th.[4)]

3. TOWARD A GENERAL EQUILIBRIUM SOLUTION

On the amount of the resources, commodities and moneys which appeared in the model, market clearing conditions are held. Therefore, after specifying the forms of the functions which also appeared in the model, we may derive general equilibrium solutions. The process to get to general equilibria is illustrated in Fig. 3.

First, as to the subjective equilibrium of household sector, variables x_i, T_i and P_i can be solved under Eqns. (8) and (10)

$$x_i = x_i(I_i, Q_i, K_i, q, r, s, \pi, \tau, \phi, \psi), \qquad (32)$$

$$T_i = T_i(I_i, \ldots\ldots\ldots\ldots, \psi), \qquad (33)$$

$$P_i = P_i(I_i, \ldots\ldots\ldots\ldots, \psi), \qquad (34)$$

and

$$t_i = t_i(I_i, \ldots\ldots\ldots\ldots, \psi). \qquad (35)$$

Secondly, as to the production sector, using Eqns. (12) and (14), resource inputs E_0 and E_1 are described

$$E_0 = E_0(\pi) \qquad (36)$$

$$E_1 = E_1(\mu, \phi) \qquad (37)$$

therefore, we may write the amount of commodity production as

$$x = f(E_0) = f(E_0(\pi)) = \hat{f}(\pi), \qquad (38)$$

$$Y = y(E_1) = y(E_1(\mu, \phi)) = \hat{y}(\mu, \phi). \qquad (39)$$

Under the market clearing conditions $\Sigma x_i = X$ and $\Sigma(t_i+T_i) = Y$, the endogeneous parameters π and ϕ may be solved from Eqns. (32), (33), (35), (38) and (39)

4) See Henderson, Quandt (1980), pp. 235–236.

$$\pi = \pi\,(q,\ r,\ s,\ \tau,\ \psi,\ \mu), \tag{40}$$
$$\phi = \phi\,(q,\ r,\ s,\ \tau,\ \psi,\ \mu). \tag{41}$$

Variables which must be determined through public sector are annual expenditures to transportation, say E_2, E_3 and E_4, travel time, say q, r and s, and prices of τ and ψ which satisfy Eqns. (23) and (25). Iteration loop is carried out until the travel time q, r and s reach equilibria.

Other given parameters are wage rate w, fuel tax rate μ and welfare weight $\partial W/\partial u^i$.

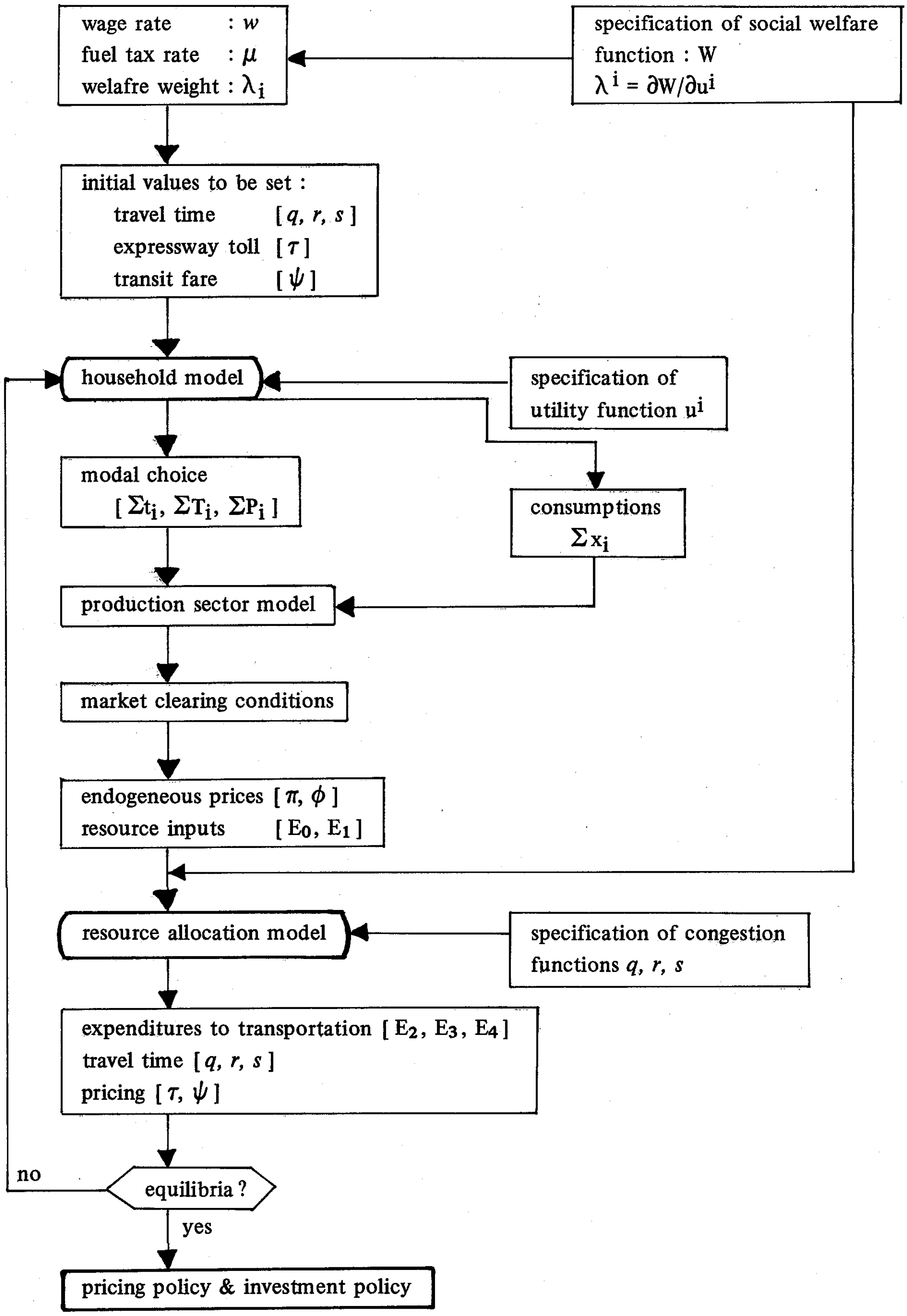


Fig. 3 : Algorism for General Equilibrium Solution

4. CONCLUDING REMARKS

This paper is a kind of preliminary study for approaching the problem of optimal investment and pricing to transportation in the frame of general equilibrium theory. We made an effort to forcus on the total structure of the model, so many unrealities and simplifications which must be urgently improved are involved in the model. Though a congestion function can be positively determined by observed data, a lot of difficulties must exist in specifying social welfare function and utility function. Moreover, trip patterns and transportation network used here are quite simplified one, but the results derived in this paper are basically important and immutable law. This study can offer effective principles to the policies of both pricing and expenditure to transportation.

As possible developments of this work in future we may point out the followings. Firstly from the standpoint of traffic engineering, (a) to take in commercial trips which are derived from industrial activities, (b) to expand the dimension of the model to urban space and transportation networks, and (c) to have a linkage to the activity-based travel demand model (e.g. Jones (1981)). Secondly, from welfare economics, (d) to test the model by comparative statics, (e) to specify the forms of social welfare function and utility function, and (f) to have a linkage to the social-surplus-maximizing approach (e.g. Hotelling (1938)).

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