Memoirs of Faculty of Engineering
Fukuyama University
No. 2, March 1980
<Technical Note>

RE-INTERPRETATION OF PRESENT PATTERN METHOD THROUGH OPTIMISING TECHNIQUES*

Katsunao KONDO**

Abstract

This note discusses the present pattern method for estimating trip distribution through optimizing techniques. It is shown that some of the conventional and experimental present pattern methods can be derived from the solutions of optimization problems , for instance, Average Factor Method from Chi-square minimizing problem, and also Detroit Model from Entropy maximizing problem.

1. Introduction

In general, as well known, a trip distribution model consists of the following two sub-models; first, the model which specifies OD pattern, X_{ij} , both theoretically and positively, and secondly the model which modifies an estimated X_{ij} so that the following two constraint equations on X_{ij} are satisfied:

$$\sum_j X_{ij} = U_i \qquad (i=1,2,\ldots\ldots,N) \qquad \ldots.(1)$$

$$\sum_i X_{ij} = V_j \qquad (j=1,2,\ldots\ldots,N) \qquad \ldots.(2)$$

---

* This note is a revision and translation of the Author's technical note, " Present Pattern Method As An Optimization Problem " (in Japanese, Proc. of JSCE, No. 272, April 1978).

** Department of Civil Engineering

where

X_{ij} is the number of trips from zone i to zone j,
U_i is the number of trips originating in zone j,
V_j is the number of trips attractig to zone i, and
N is the total number of zones in a study area.

So-called gravity model has been often used as a first sub-model with the Growth Factor Technique (Present Pattern Method) which acts as a second sub-model. Let $\overline{X}_{ij}$ be the first-stage approximation of a first sub-model, and X_{ij} be the final solution of a second sub-model. Then, we can define balancing factor f_{ij} for the (i, j) th pair of OD matrix as

$$X_{ij} = f_{ij}\ \overline{X}_{ij}. \qquad ....(3)$$

The values of balancing factors f_{ij}'s are determined through equations (1) and (2) after specifying the form of f_{ij}. If X_{ij} is an observed OD matrix in the base year, this calculation process of determining f_{ij} is called " Present Pattern Method ". In this case, balancing factor f_{ij} is equivalent to the growth factor for the (i, j) th pair which reflects the growth in trip distributions expected between the base and horizon years.

On the other hand, there exist many kinds of Present Pattern Method. They can be classified into two types. One is the conventional Growth Factor Technique which includes Detroit Model, Average Factor Method, Frator Method, etc. The other is an optimising technique. The former is an empirical model, and we can reach the final solutions through the process of iterative calculation. On the contrary, the latter is a theoretical model, and the form of f_{ij} can be specified by optimising particular objective function under the constraint equations (1) and (2).

In 1962, Tsukahara[1] has proposed a " Residual Sum of Squares Minimising Model " (RSS-Minimising Model). The philosophy of the RSS-Minimising Model is that the future OD pattern should be "most similar" to the present OD pattern. Few researcher has given a thought to the philosophy of such kind before and after Tsukahara's model. Then, this paper firstly reviews Tsukahara's model in a more ėlegant form, and secondly introduces some alternative optimising techniques which offer the theoretical basis to some of the conventional Present Pattern Methods.

2. Residual Sum of Squares Minimizing Model

Tsukahara's RSS-Minimising Model was derived from following two conditions:

(i) equations (1) and (2) should be perfectly satisfied, and

(ii) future OD matrix should be determined so that its OD pattern forms the most similar one to the present OD pattern.

If the condition (ii) does not exist, we can make an infinite number of OD matrices which satisfy the equations (1) and (2). The condition (ii), therefore, makes it possible to select one matrix among an infinite number of OD matrices through giving some definition to the concept of the "most similar" pattern. The concept of the "most similar pattern" adopted in his model was to minimise the residual sum of squares between the future OD pattern and the present OD pattern. Here we use a "unit OD matrix" as an OD pattern, then his problem can be formulated as

$$\text{minimise} \quad \sum_{ij} (X_{ij}/T - \bar{P}_{ij})^2$$

$$\text{subject to} \quad \sum_{j} X_{ij} = U_i, \quad (i=1,2,\ldots,N) \qquad \ldots.(4)$$

$$\sum_{i} X_{ij} = V_j. \quad (j=1,2,\ldots,N) \qquad \ldots.(5)$$

where

T is the total number of trip generations in the horizon year, and $\bar{P}_{ij}$ is a unit OD matrix of the base year.

$$T = \sum_{ij} X_{ij} \quad : \text{given}$$

$$\bar{P}_{ij} = \bar{X}_{ij} / \sum_{ij} \bar{X}_{ij} = \bar{X}_{ij}/\bar{T} \text{ : given}$$

(The notation $\bar{X}$ identifies the present value of X.)
This problem can be solved by a Lagrange formulation.
Let

$$L = \sum_{ij} (X_{ij}/T - \bar{P}_{ij})^2 + \sum_i \alpha_i (U_i - \sum_j X_{ij})$$
$$+ \sum_j \beta_j (V_j - \sum_i X_{ij}) \quad \ldots(6)$$

where
the α_i's are the Lagrange ultipliers associated with the set of equations (4), and the β_j's those associated with (5). Then optimality conditions can be stated as

$$\partial L/ \alpha_i = U_i - \sum_j X_{ij} = 0 \quad \ldots.(7)$$
$$\partial L/\partial\beta_j = V_j - \sum_i X_{ij} = 0 \quad \ldots..(8)$$

and then

$$\partial L/\partial X_{ij} = 2(X_{ij} - T \bar{P}_{ij})/ T^2 - \alpha_i - \beta_j = 0 \quad \ldots..(9)$$

Equations (7) and (8) are, as usual, repetitions of the constraint equations. From equation (9), we can obtain an expression for X_{ij} in terms of the multipliers as

$$X_{ij} = T \bar{P}_{ij} + \frac{T^2}{2} (\alpha_i + \beta_j). \quad \ldots..(10)$$

For convenience, we use instead variables λ_i and μ_j defined by

$$\lambda_i = \alpha_i T^2/2 \quad \text{and} \quad \mu_j = \beta_j T^2/2.$$

Then, the main equation (10) can be written as

$$X_{ij} = T\,\overline{P}_{ij} + (\lambda_i + \mu_j). \qquad .....(11)$$

The second term in the right hand of the equation (11), $\lambda_i + \mu_j$, represents the residual between future OD traffic and the present pattern of OD traffic, $\overline{T}P_{ij}$. This residual, $\lambda_i + \mu_j$, must be determined by substitution in equations (7) and (8). Fortunately we can obtain the value of $\lambda_i + \mu_j$ in a positive form as

$$\lambda_i + \mu_j = \frac{1}{N}(U_i + V_j) - T\left(\sum_k \overline{P}_{ik} + \sum_k \overline{P}_{kj}\right) \qquad ....(12)$$

As evident from equation (12), the value of $\lambda_i + \mu_j$ is unique to the (i, j) th pair, however it is not possible to determine the unique value of each λ_i or μ_j. Having equation (12), however, we can calculate future OD traffic by equation (11). It should be noted that unique solutions of λ_i and μ_j are not required to our purpose.

Above is an outline of the Tsukahara's RSS-Minimising Model in a more elegant form than the original one. However, some improvements will be required to the concept of the "most similar patterns", say "minimising the residual sum of squares", because the least squares of residuals is not necessarily a preferrable concept for the "most similar patterns". Therefore we introduce other alternative concept for it below.

3. Chi-Square Minimizing Model

In RSS-Minimising Model, the most similar pattern was thought to be a pattern which minimises the sum of squares of deviations between the future unit OD matrix given by X_{ij}/T and the present unit OD matrix given by $\overline{P}_{ij}$. In another words, "absolute error" was evaluated as deviations between the future and the present. In order to evaluate the relative error, here we consider "Chi-square" as a measure for deviation between the fure and the present.

Chi-square which should be minimised is defined by

$$\chi^2 = \sum_{ij} \frac{(T\bar{P}_{ij} - X_{ij})^2}{T\bar{P}_{ij}} \qquad \text{......(13)}$$

where

T is the total number of trip generations in a study area in horizon year, and $\bar{P}_{ij}$ is a probability that a randomly selected observation falls in the (i, j) th cell of OD matrix. $\bar{P}_{ij}$ is nothing but the present unit OD matrix. $T\bar{P}_{ij}$ is, therefore, the expected number of trips between zones i and j in the horizon year.

This χ^2 -minimising model can be solved also by a Lagrangian formulation under the constraint equations (1) and (2).

Let

$$L = \chi^2 + \sum_i \alpha_i (U_i - \sum_j X_{ij}) + \sum_j \beta_j (V_j - \sum_i X_{ij}) \qquad \text{.....(14)}$$

where

the α_i 's and β_j 's are Lagrange multipliers associated with the set of equations (1) and (2). From $\partial L / \partial X_{ij} = 0$, we can obtain an expression for each X_{ij} in terms of the multipliers as

$$X_{ij} = \frac{\alpha_i + \beta_j}{2} \bar{P}_{ij} \ . \qquad \text{.....(15)}$$

Some readers must have noticed that this form of solution is quite similar to that of the Average Growth Factor Technique, whose first approximation is expressed by

$$X_{ij}^{(1)} = \frac{F_i^{(1)} + G_j^{(1)}}{2} \bar{X}_{ij} \qquad \text{.....(16)}$$

where

$F_i^{(1)}$ and $G_j^{(1)}$ are the growth factors for zones i and j which reflect the growth in trip productions and trip attractions expected between the base and horizon years.

$$F_i^{(1)} = U_i / \bar{U}_i \quad \text{and} \quad G_j^{(1)} = V_j / \bar{V}_j \qquad \text{.....(17)}$$

However it should be noted that the solution X_{ij} in equation (15) is not necessarily equal to $X_{ij}^{(1)}$ in equation (16), because of the difference of calculating process for satisfying the constraint equations (1) and (2). But, at least, we can say that χ^2-minimising model offers a theoretical basis to the first approximation $X_{ij}^{(1)}$ in equation (16) of the Average Growth Factor Technique.

For convenience, in equation (15), we use instead variables λ_i and μ_j defined as

$$\lambda_i = \alpha_i / 2\bar{T} \quad \text{and} \quad \mu_j = \beta_j / 2\bar{T} .$$

Then the equation (15) can be rewritten as

$$X_{ij} = (\lambda_i + \mu_j) \bar{T} \bar{P}_{ij}$$

$$= (\lambda_i + \mu_j) \bar{X}_{ij} \qquad \text{.....(18)}$$

This is the final form of solution of χ^2-minimising model. Needless to say, $\lambda_i + \mu_j$ must be determined through equations (1) and (2). However we can not exclude the possibility that $\lambda_i + \mu_j$ will take nagative value. Consequently, Average Growth Factor Technique is one of the ways that avoid the occurrence of negative solution. In that case, the final OD patterns fairly differ from the solutions which minimise χ^2 in equation (13).

4. Entropy Maximizing Model As Present Pattern Method

In Growth Factor Technique, the number of trips between zones

i and j is estimated only by use of both giben OD table of the base year and the growth factors for zones i and j. Here, we give a new interpretation to the Present Pattern Method by preparing following two conditions:

(i) constraint equations (1) and (2) on X_{ij} should be perfectly satisfied, and

(ii) future OD matrix should be defined so that its pattern becomes "the most probable" one under the given a priori probabilities.

As to condition (ii), the concept of "the most probable" means to maximise the joint probability given by

$$P = \frac{T!}{\prod_{ij}(X_{ij}!)} \prod_{ij} (\overline{P}_{ij})^{X_{ij}} \qquad \text{.....(19)}$$

where $\overline{P}_{ij}$ is a priori probability that a randomly selected observation falls into the (i, j) th cell of future OD matrix. Needless to say, $\overline{P}_{ij}$ is assumed to be equal to the present unit OD matrix. Then our problem is expressed as follows:

maximise P

subject to equations (1) and (2).

To maximise P is equivalent to maximising ln P. Therefore, our problem can be rewritten as follows by using Lagrange multipliers, α_i's and β_j's, after omitting constant terms and using Stirling's approximation $(\ln X! = X \ln X - X)$:

maximise L,

$$L = \sum_{ij} X_{ij} \ln \overline{P}_{ij} - \sum_{ij} X_{ij} \ln X_{ij} + \sum_i \alpha_i (\sum_j X_{ij} - U_i) + \sum_j \mu_j (\sum_i X_{ij} - V_j). \qquad \text{.....(20)}$$

The X_{ij} 's which maximise L, and which therefore constitute the most probable distribution of trips, are the solutions of

$$\partial L / \partial X_{ij} = 0 \qquad .....(21)$$

and the constraint equations (1) and (2). Equation (21) gives us

$$X_{ij} = \exp(\alpha_i + \beta_j - 1)\ \bar{P}_{ij} \ . \qquad .....(22)$$

To obtain the final result in more familiar form, write

$$\lambda_i = \exp(\alpha_i)/\ \bar{T}, \quad \mu_j = \exp(\beta_j - 1)$$

and then we have

$$X_{ij} = \lambda_i \mu_j\ \bar{T}\ \bar{P}_{ij} = \lambda_i \mu_j\ \bar{X}_{ij} \qquad .....(23)$$

where λ_i and μ_j are balancing factors. These balancing factors must be determined so that the following constraint equations are satisfied:

$$\sum_j X_{ij} = \lambda_i \sum_j \mu_j \bar{X}_{ij} = U_i \ , \qquad .....(24)$$

$$\sum_i X_{ij} = \mu_j \sum_i \lambda_i \bar{X}_{ij} = V_j \ . \qquad .....(25)$$

From equations (24) and (25), we have

$$\lambda_i = U_i / \sum_j \mu_j \bar{X}_{ij} \qquad (i=1,2,\ldots,N) \qquad .....(26)$$

$$\mu_j = V_j / \sum_i \lambda_i \bar{X}_{ij} \qquad (j=1,2,\ldots,N) \qquad .....(27)$$

These non-linear equations on λ_i's and μ_j's can be solved by iteration method, but here we use another simple way to solve them. In order to eliminate λ_i's, we substitute following equations (28) and (29) for equations (23), (26) and (27). Then we have

$$X_{ij} = \frac{\mu_j \overline{X}_{ij}}{\sum_k \mu_k \overline{X}_{ik}} U_i \qquad .....(28)$$

$$\mu_j = V_j \Big/ \sum_i \left[\frac{U_i \overline{X}_{ij}}{\sum_k \mu_k \overline{X}_{ik}} \right] \quad (j=1,2,...,N) \qquad ....(29)$$

This form of solution is more familiar to us. Our initial form of solution expressed in equation (23) had been doubly-constrained, however now we obtained singly-constrained solution as above.

Thus the most probable distribution of trips is expressed by using balancing factors μ_j's. And equation (28) is the final form of the solutions of Joint Probability Maximising Model. We give the name of "Entropy-Maximising Distribution Model of Present Pattern Type" to this new technique, because the second term of the equation (20), $-\sum X_{ij} \ln X_{ij}$, is called "Entropy" in the field of Information Theory.

In addition, we can prove that the solutions of Entropy Maximising Model are completely equivalent to those of Detroit Model. Shortly speaking, the process for solving equations (26) and (27) is equivalent to that of iteration in Detroit Model Which can be expressed by

$$X_{ij}^{(1)} = \overline{X}_{ij} F_i G_j / H \qquad .....(30)$$

where

$\overline{X}_{ij}$: the base-year OD matrix,
F_i : the growth factor for trips produced in zone i,
G_j : the growth factor for trips attracted to zone j,

H : the growth factor for trips generated in a study area,

$X_{ij}^{(1)}$: the first approximation fo the horizon-year OD matrix.

Iteration must be carried out until the values of the following recalculated growth factors approach nearly "one".

$$F_i^{(1)} = U_i / \sum_j X_{ij}^{(1)} , \qquad \text{......(31)}$$

$$G_j^{(1)} = V_j / \sum_i X_{ij}^{(1)} . \qquad \text{......(32)}$$

Essentially the growth factor H is not important in Equation (30), because it is constant. Then we ommit H in the following discussion.

From Equation (30), (31) and (32), we have

$$F_i F_i^{(1)} = U_i / \sum_j \overline{X}_{ij} G_j , \qquad \text{.......(33)}$$

$$G_j G_j^{(1)} = V_j / \sum_i \overline{X}_{ij} F_i . \qquad \text{.......(34)}$$

If iteration ends at the n-th step, this iteration procedure can be written as follows:

$$F_i F_i^{(1)} F_i^{(2)} \ldots F_i^{(n)} = U_i / \sum_j \overline{X}_{ij} (G_j G_j^{(1)} G_j^{(2)} \ldots . G_j^{(n)}) , \text{..(35)}$$

$$G_j G_j^{(1)} G_j^{(2)} \ldots G_j^{(n)} = V_j / \sum_i \overline{X}_{ij} (F_i F_i^{(1)} F_i^{(2)} \ldots . F_i^{(n)}) \text{...(36)}$$

Compare these equations (35) and (36) to equations (26) and (27), say

$$\lambda_i = U_i / \sum_j \overline{X}_{ij} \mu_j ,$$
$$\mu_j = V_j / \sum_i \overline{X}_{ij} \lambda_i .$$

We can expect that both iterations reach same solutions because of the similarity in their equations. According to our calculation, as a matter of facts, both iterations reached the same solutions. In another words, we can express this result as

$$\lambda_i = F_i F_i^{(1)} F_i^{(2)} \ldots\ldots F_i^{(n)} ,$$
$$\mu_j = G_j G_j^{(1)} G_j^{(2)} \ldots\ldots G_j^{(n)} .$$

Only difference between Entropy model and Detroit model lies in the technique for convergence criterion. Entropy-maximising model as present pattern method, therefore, gives the theoretical basis to Detroit model.

5. Summary

In this paper, we could give a statistical interpretation to some of the conventional Present Pattern Methods. This result is summarised in Table 1.

There is no doubt that a better understanding of the relationship between the conventional Present Pattern Method and the optimisation problem would be of assistance to planners. But more studies are required in this field, and more statistical optimisation problems should be considered.

Table 1 : Relationship between conventional Present Pattern Methods and optimisation problems

Model	Structure of Solution	Note
RSS-minimising model (Least Squares model)	$X_{ij} = TP_{ij} + \lambda_i + \mu_j$	$\lambda_i + \mu_j = \frac{1}{N}(U_i + V_j) - T(\sum_k \overline{P}_{ik} + \sum_k \overline{P}_{kj})$
Chi-square minimising model	$X_{ij} = (\lambda_i + \mu_j)\overline{X}_{ij}$	1. This form of solution is similar to that of the first approximation in Average Factor Method. 2. Non-negativity of the solutions is not necessarily guaranteed.
Entropy maximising model	$X_{ij} = \lambda_i \mu_j \overline{X}_{ij} = \frac{\mu_j \overline{X}_{ij}}{\sum_k \mu_k \overline{X}_{ik}} U_i$	This model offers theoretical basis to Detroit model.

6. Reference

[1] Tsukahara, S. : On a matrix distribution, Transportation and Economics, Vol. 22, No. 2, 1962. (in Japanese)