ARMA MODELS WITH INVERSE GAUSSIAN INNOVATIONS

Masanori B. Okamoto

Dept.Economics, Fukuyama University

1. INTRODUCTION

Most of the theory behind the analysis of time series are based on the underlying assumption that the innovations are normally distributed. However time series processes driven by non-Gaussian are common in real situations. Nelson and Granger(1979) reported that among 21 time series of economic variables, 17 time series have non-Gaussian innovations. Recently non-Gaussian distribution of financial daily return has shown by Taylor(1986), Kariya and Matsue(1988). Theoretical contributions to time series with non Gaussian innovations have been done by Davies et al(1980), Li and McLeod(1988) and Damsleth and El-Shaarawi(1989).

In this report skewness and kurtosis of $\{X_t\}$ which follow ARMA(p,q) models with inverse Gaussian innovations are given. It is show that MA models with inverse Gaussian inovations are preserved in the form of MA models with chi-squared innovations which are induced from the original inverse Gaussian variable. The regularity conditions proposed by Li and McLeod(1988) for the conditional maximum likelihood estimator with non-Gaussian innovations are verified for the

AR process with chi-squared innovations. In this case we can not expect the existence of conditional maximum likelihood estimator for the degree of freedom of chi-squared distribution less than and equal to 2. The transformed MA model is the same. The regularity conditions also are checked for the power inverse Gaussian and the inverse Gaussian innovations.

2. INVERSE GAUSSIAN INNOVATIONS

Let stationary time series $\{X_t, t=1,2,...\}$ follow a ARMA (p,q) model

$$\phi(B)X_t = \theta(B)a_t \tag{1}$$

where B is a shift-operator and the roots of polynomials $\phi(B)=1-\phi_1 B-\phi_2 B^2...-\phi_p B^p$ and $\theta(B)=1-\theta_1 B-\theta_2 B^2...-\theta_q B^q$ lie outside the unit circle and assume that there are no common roots among them. Now a_t follows a i.i.d. inverse Gaussian. We here adopt Iwase's expression of this distribution(e.g.Iwase and Hirano(1990)),

$$f(a_t)da_t = \frac{1}{\sqrt{2\pi c}}\left(\frac{a_t}{\mu}\right)^{-\frac{3}{2}} \exp\left\{-\frac{1}{2}\left(\frac{\sqrt{a_t/\mu}-\sqrt{\mu/a_t}}{c}\right)^2\right\}\frac{da_t}{\mu} \tag{2}$$

where $0<a_t, \mu, c<\infty$. This expression is related to that given by Chhikara and Folks(1989) with $c^2=\mu/\lambda$.

Now we obtain the MA model with infinite order from the ARMA model assuming of its invertibility,

$$X_t = \phi^{-1}(B)\theta(B)a_t = \psi(B)a_t$$

where $\psi(B)=1+\psi_1 B+\psi_2 B^2+...$ Skewness $\sqrt{\beta_1(X)}$ and kurtosis $\beta_2(X)$ are obtained as in Davies, Spedding and Watoson(1980),

$$\sqrt{\beta_1(X)} = \frac{\sum_{i=0}^{\infty} \psi_i^3 \sqrt{\beta_1(a_t)}}{(\sum_{i=0}^{\infty} \psi_i^2)^{3/2}} = \frac{\sum_{i=0}^{\infty} \psi_i^3}{(\sum_{i=0}^{\infty} \psi_i^2)^{3/2}} 3c, \quad (4)$$

$$\beta_2(X) = \frac{\sum_{i=0}^{\infty} \psi_i^4}{(\sum_{i=0}^{\infty} \psi_i^2)^2}(15c^2+3) + 6\frac{\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \psi_i \psi_j}{(\sum_{i=0}^{\infty} \psi_i^2)^2} \quad (5)$$

Note that skewness $\sqrt{\beta_1(a_t)}=3c$ and kurtosis $\beta_2(a_t)=15c^2+3$ for inverse Gaussian. When $c \to 0$, inverse Gaussian distribution approaches to Gaussian and the distribution of X_t approaches to the one in Gaussian case.

3. TRANSFORMED MA MODEL

When at follows IG(μ ,c^2), the transformed innovations

$$\varepsilon_t=\{(a_t/\mu)^{1/2}-(\mu/a_t)^{1/2}\}^2/c^2 \quad (6)$$

follow $\chi^2(1)$ distribution. Assuming the existence of $\psi(B)$ for $X_t=\psi(B)a_t$, we obtain $Y_t=\psi(B)\varepsilon_t$ for the transformed variable Y_t from X_t such that

$$Y_t=\{(X_t/\mu)^{1/2}-(\mu/X_t)^{1/2}\}^2/c^2. \quad (7)$$

Therefore the MA model with inverse Gaussian innovations can be preserved in the from of the MA model with the same coefficient but with the transformed variable y_t.

<u>Theorem 1</u> : An invertible stationary MA model $X_t=\psi(B)a_t$ of infinite order with inverse Gaussian innovations i.e., $a_t \sim IG(\mu .c^2)$, can be also transformed into the MA model

$$Y_t = \psi(B)\varepsilon_t \tag{8}$$

with the same $\psi(B)$, where ε_t is given by (6) and Y_t is given by (7).

(Proof) $\varepsilon_t = \{(a_t/\mu)^{1/2} - (\mu/a_t)^{1/2}\}^2/c^2 = a_t(a_t-\mu)^2/(\mu a_t^2 c^2)$

$$= \psi^{-1}(B)\psi(B)a_t\{\frac{\psi(B)a_t - \mu}{\mu^{1/2}\psi(B)a_t c}\}^2$$

$$= \psi^{-1}(B)[\{(\psi(B)a_t)/\mu\}^{1/2} - \{\mu/(\psi(B)a_t)\}^{1/2}]^2/c^2 = \psi^{-1}(B)Y_t.$$

4. CONDITIONAL MAXIMUM LIKELIHOOD ESTIMATOR

Li and McLeod(1988) give the regularity conditions for the existence of conditional maximum-likelihood estimator in the ARMA model with non-Gaussian innovations and its consistency. Denote the probability density function of a_t by $f(a_t) \equiv f(a_t | \alpha)$ where α is the parameter of its density.

Condition 1. The derivatives

$$\frac{\partial \ln f(a_t)}{\partial \alpha}, \frac{\partial \ln f(a_t)}{\partial a_t}, \frac{\partial^2 \ln f(a_t)}{\partial \alpha^2}, \frac{\partial^2 \ln f(a_t)}{\partial a_t \partial \alpha}, \frac{\partial^2 \ln f(a_t)}{\partial a_t^2} \tag{9}$$

exist, and are continuous for almost all a_t and α in an open interval of α including the true value.

Condition 2. At the true values of parameters

$$E\{\frac{1}{f(a_t)}\frac{\partial f(a_t)}{\partial \alpha}\} = E\{\frac{1}{f(a_t)}\frac{\partial^2 f(a_t)}{\partial \alpha^2}\} = 0 \tag{10}$$

and

$$E\{\frac{1}{f(a_t)^2}(\frac{\partial f(a_t)}{\partial \alpha})^2\} > 0, \tag{11}$$

and similarly

$$E\{\frac{1}{f(a_t)}\frac{\partial^2 f(a_t)}{\partial \alpha\ \partial a_t}\} = 0,\quad 0 < E\{(\frac{1}{f(a_t)}\frac{\partial f(a_t)}{\partial a_t})^2\} < \infty \tag{12}$$

Condition 3.

$$\lim_{a_t \to a} f(a_t) = \lim_{a_t \to b} f(a_t), \ \lim_{a_t \to a} \frac{\partial f(a_t)}{\partial a_t} = \lim_{a_t \to b} \frac{\partial f(a_t)}{\partial a_t} \quad , \tag{13}$$

where the open interval (a,b) is the range of a_t.

They reported that innovations of Gamma distribution Ga(α,1) satisfied all conditions for $\alpha > 2$. Similarly the innovations a_t with chi-squared distribution of the degree of freedom 2α

$$f(a_t) = \frac{1}{2^{\alpha}\Gamma(\alpha)} a_t^{\alpha-1} \exp(-\frac{a_t}{2}), \quad a_t > 0, \tag{14}$$

in $\phi(B)X_t = a_t$ satisfy all conditions for $\alpha > 2$. In fact, for $\alpha > 2$

$$E\{(\frac{1}{f(a_t)} \frac{\partial f(a_t)}{\partial a_t})^2\} = \frac{1}{4(\alpha-2)} > 0. \tag{15}$$

In this case the log-likelihood L conditional on the first p observations is given by

$$L = \sum_{t=p+1}^{n} (\alpha-1) \ln a_t - \frac{1}{2} \sum_{t=p+1}^{n} a_t - (n-p)\{\alpha \ln 2 + \ln \Gamma(\alpha)\}. \tag{16}$$

However when innovations a_t follow $\chi^2(1)$ (i.e., $\alpha = 1/2$), the value of (15) is $-(1/6) < 0$. Thus within the condition 2, only the condition

$$0 < E\{(\frac{1}{f(a_t)} \frac{\partial f(a_t)}{\partial (a_t)})^2\} < \infty \tag{17}$$

does not hold for $\alpha = 1/2$. The condition 3 is also not satisfied for $\alpha = 1/2$. In fact

$$\lim_{a_t \to 0} f(a_t) = \infty, \qquad \lim_{a_t \to \infty} f(a_t) = 0. \tag{18}$$

Therefore we do not expect the existence of conditional maximum-likelihood estimator for the AR model with chi-squared innovations of the degree of freedom 1 as well as the transformed MA model (8).

5. POWER INVERSE GAUSSIAN INNOVATIONS

Next suppose that the innovations a_t in ARMA(p,q)

$$\phi(B)X_t = \theta(B)a_t \tag{19}$$

are distributed as the power inverse Gaussian distribution, of which probability densities are given by Iwase and Hirano (1990) as $f(a_t;\mu,c^2,\lambda)=$

$$\frac{1}{\sqrt{2\pi c}}\left(\frac{a_t}{\mu}\right)^{-1-\frac{\lambda}{2}}\exp\left\{-\frac{1}{2(\lambda c)^2}\left(\left(\frac{a_t}{\mu}\right)^{\frac{\lambda}{2}}-\left(\frac{a_t}{\mu}\right)^{-\frac{\lambda}{2}}\right)^2\right\}\frac{1}{\mu} \tag{20}$$

where $a_t>0$, $\mu>0$, $c>0$, $\infty>\lambda>-\infty$. This distribution is reduced to inverse Gaussian when $\lambda=1$ and to lognormal distribution when $\lambda\to 0$. Let denote parameter vector $\eta'=(\zeta',\alpha')$ where $\zeta'=(\phi_1,\ldots,\phi_p,\theta_1,\ldots,\theta_p)$ and $\alpha'=(\mu,c^2,\lambda)$. Li and McLeod's condition 1 and condition 3 are all satisfied, but the condition 2 are partly satisfied. In fact,

$$E\left\{\frac{\partial \ln f(a_t)}{\partial \mu}\right\}=0,\quad E\left\{\frac{\partial \ln f(a_t)}{\partial c^2}\right\}=0, \tag{21}$$

and

$$E\left\{\left(\frac{\partial \ln f(a_t)}{\partial \mu}\right)^2\right\}=\left(\frac{\lambda}{\mu}\right)^2\left\{\frac{1}{2}+\left(\frac{1}{(\lambda c)}\right)^2\right\}>0. \tag{22}$$

However

$$E\left\{\left(\frac{\partial \ln f(a_t)}{\partial c^2}\right)^2\right\}=\frac{c^4(6-c^2)+3}{4c^8}>0 \tag{23}$$

holds only for $c^2\leqq 6$. Further using the formula given by Iwase (1991) (see

Appendix I) we have

$$E\{\frac{\partial \ln f(a_t)}{\partial \lambda}\}=\frac{1}{\lambda}(1-\frac{1}{2c^2})=0. \qquad (24)$$

only for $c^2=1/2$. And more

$$E\{\frac{1}{f(a_t)}\frac{\partial^2 f(a_t)}{\partial (c^2)^2}\}=0. \qquad (25)$$

The expressions of

$$E\{\frac{1}{f(a_t)}\frac{\partial^2 f(a_t)}{\partial \mu \partial a_t}\}, \qquad E\{\frac{1}{f(a_t)}\frac{\partial^2 f(a_t)}{\partial c^2 \partial a_t}\}, \qquad (26)$$

$$E\{\frac{1}{f(a_t)}\frac{\partial^2 f(a_t)}{\partial \mu^2}\}, \qquad E\{\frac{1}{f(a_t)}\frac{\partial^2 f(a_t)}{\partial a_t}\}, \qquad (27)$$

are given in Appendix II as polynomials of the modified Bessel functions of the second kind and parameters $\alpha' =(\mu, c^2, \lambda)$.

Particularly for the case $\lambda=1$ (inverse Gaussian) we have (21) and (22) as the first order derivatives with μ and c^2. Expectations of the second order derivatives are zero within some range or at a particular value of c^2 as follows;

$$E\{\frac{1}{f(a_t)}\frac{\partial^2 f(a_t)}{\partial \mu \partial a_t}\}=\frac{1}{4c^2\mu^2}(1+c^2-3c^4-6c^2)=0 \qquad (28)$$

for $c^2=1/2$,

$$E\{\frac{1}{f(a_t)}\frac{\partial^2 f(a_t)}{\partial c^2 \partial a_t}\}=\frac{3}{4c^2\mu}(2c^4-1)=0 \qquad (29)$$

for $c^4=1/2$,

$$E\{\frac{1}{f(a_t)}\frac{\partial^2 f(a_t)}{\partial \mu^2}\}=\frac{1}{c^2\mu^2}(1+c^2)>0. \qquad (30)$$

Finally

$$E\{(\frac{1}{f(a_t)}\frac{\partial f(a_t)}{\partial a_t})^2\}=\frac{3(-21c^8-16c^6-c^4+3c^2+1)}{4c^4\mu^2}<+\infty \quad (31)$$

holds only for c^2 such that $0<c^2<0.4384$.

For the existence and weak consistency of L.M.E. of $\eta=(\phi,\sigma,\alpha)'$ we now expand $n^{-1}\frac{\partial L}{\partial \eta}$ about the true parameter $\overline{\eta}$ and evaluating at η,

$$n^{-1}\frac{\partial L}{\partial \eta}=n^{-1}\frac{\partial L}{\partial \eta}\Big|_{\eta=\overline{\eta}}+n^{-1}\frac{\partial^2 L}{\partial \eta\,\partial \eta'}\Big|_{\eta=\eta^*}(\eta-\overline{\eta}) \quad (32)$$

where η^* lies between η and $\overline{\eta}$. The components of the first term are given as follows,

$$n^{-1}\frac{\partial L}{\partial \phi_i}\xrightarrow{p}0,\quad n^{-1}\frac{\partial L}{\partial \theta_i}\xrightarrow{p}0,\quad \text{(Li and Mcleod, 1988)}$$

$$n^{-1}\frac{\partial L}{\partial c2}=n^{-1}\Sigma\frac{1}{f(a_t)}\frac{\partial f(a_t)}{\partial c^2}\xrightarrow{p}E\{\frac{1}{f(a_t)}\frac{\partial f(a_t)}{\partial c^2}\}=0,$$

$$n^{-1}\frac{\partial L}{\partial \lambda}=n^{-1}\Sigma\frac{1}{f(a_t)}\frac{\partial f(a_t)}{\partial \lambda}\xrightarrow{p}E\{\frac{1}{f(a_t)}\frac{\partial f(a_t)}{\partial \lambda}\}=\frac{1}{\lambda}(1-\frac{1}{2c^2})\neq 0.$$

Thus for inverse Gaussian case ($\lambda=1$), the first term of (30) converges stochastically to zero with variance $-n^{-1}$\{expected value of $n^{-1}\frac{\partial^2 L}{\partial \eta\,\partial \eta'}$\}. As in Crowder(1976) it is shown that the conditional M.L.E. $\hat{\eta}$ of η exists and it is consistent. Fisher information matrix contains some elements including a constant which depends on c^2.

6. CONCUSION

Although an invertible stationary MA model of infinite order with inverse Gaussian innovations can be transformed into the MA model with innovations which follow chi-squared distribution of freedom 1 under the conditions of invertbility. Conditional maximum likelihood estimator for this model with chi-squared innovations is not exist. When innovations in ARMA(p.q) are distributed as power inverse Gaussian which includes inverse Gaussian, conditional maximum lilelihood estimator exists for restricted values of parameter c^2, but conditional M.L.E. asymptotically exists and is consistent. Fisher information matrix contains some elements including a constant which depends on c^2.

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Appendix I

In the course of obtaining (24) we use the formula given by Iwase(private communication)

$$E[\{(\frac{a_t}{\mu})^{-\lambda}+1\}\log(\frac{a_t}{\mu})^{-\lambda}]=1-(\lambda c)^2\exp(\frac{2}{(\lambda c)^2})E_i(-\frac{2}{(\lambda c)^2})$$

where $E_i(t)=\int_{-\infty}^{t}x^{-1}e^{x}dx.$

Appendix II

The expressions of (26) and (27) are obtained from the following equations

$$\frac{1}{f(a_t)}\frac{\partial^2 f(a_t)}{\partial\mu\,\partial a_t}=\frac{\partial^2\ln f(a_t)}{\partial\mu\,\partial a_t}+\frac{\partial\ln f(a_t)}{\partial\mu}\frac{\partial\ln f(a_t)}{\partial a_t},$$

$$\frac{\partial^2\ln f(a_t)}{\partial\mu\,\partial a_t}=\frac{1}{2c^2\mu^2}\{(\frac{a_t}{\mu})^{\lambda-1}+(\frac{a_t}{\mu})^{-\lambda-1}\},$$

$$\frac{\partial\ln f(a_t)}{\partial\mu}=\frac{\lambda}{2\mu}+\frac{1}{2(\lambda c)}(\frac{\lambda}{\mu})\{(\frac{a_t}{\mu})^{\lambda}-(\frac{a_t}{\mu})^{-\lambda}\},$$

$$\frac{\partial\ln f(a_t)}{\partial a_t}=-(1+\frac{\lambda}{2})\frac{1}{a_t}-\frac{1}{2(\lambda c)^2}(\frac{\lambda}{\mu})\{(\frac{a_t}{\mu})^{\lambda-1}-(\frac{a_t}{\mu})^{-\lambda-1}\}.$$

Thus we have

$$E(\frac{1}{f(a_t)}\frac{\partial^2 f(a_t)}{\partial\mu\,\partial a_t})=\{\frac{1}{2(\lambda c)^4}(\frac{\lambda}{\mu})^2-\frac{\lambda}{2\mu^2}(1+\frac{\lambda}{2})\}E(\frac{a_t}{\mu})^{-1}$$

$$+\frac{1}{2(\lambda c)^2}(\frac{\lambda}{\mu})^2E\{(\frac{a_t}{\mu})^{\lambda-1}+(\frac{a_t}{\mu})^{-\lambda-1}\}-\{(1+\frac{\lambda}{2})\frac{1}{2(\lambda c)^2}$$

$$(\frac{\lambda}{\mu})\frac{1}{\mu}+\frac{1}{4(\lambda c)^2}(\frac{\lambda}{\mu})^2\}E\{(\frac{a_t}{\mu})^{\lambda-1}+(\frac{a_t}{\mu})^{-\lambda-1}\}$$

$$-\frac{1}{4(\lambda c)^4}(\frac{\lambda}{\mu})^2E\{(\frac{a_t}{\mu})^{2\lambda-1}+(\frac{a_t}{\mu})^{-2\lambda-1}\}$$

$$= \frac{1}{4\lambda^3 c^5 \mu^2} \sqrt{(2/\pi)} \exp\left(\frac{1}{(\lambda c)^2}\right) [2(\lambda c)^2 \{K_{1/\lambda - 1/2}\left(\frac{1}{\lambda^2 c^2}\right) +$$

$$K_{1/\lambda + 3/2}\left(\frac{1}{\lambda^2 c^2}\right)\} - \{K_{1/\lambda - 3/2}\left(\frac{1}{\lambda^2 c^2}\right) + K_{1/\lambda + 5/2}\left(\frac{1}{\lambda^2 c^2}\right)\}]$$

$$+ \frac{\lambda}{2c\mu^2}\left(1 + \frac{1}{\lambda}\right)\left(1 + \frac{2}{\lambda}\right) \sqrt{(2/\pi)} \exp\left(\frac{1}{\lambda^2 c^2}\right) K_{1/\lambda + 1/2}\left(\frac{1}{\lambda^2 c^2}\right)$$

$$+ \frac{1}{2\lambda^2 c^4 \mu^2}\left(1 - \lambda^3 c^4 - \frac{1}{2}\lambda^4 c^4\right)(1 + c^2).$$

Similarly from the previous equations and the followings

$$\frac{\partial \ln f(a_t)}{\partial c^2} = -\frac{1}{2c^2} + \frac{1}{2\lambda^2 c^4}\left\{\left(\frac{a_t}{\mu}\right)^{\lambda/2} - \left(\frac{a_t}{\mu}\right)^{-\lambda/2}\right\}^2,$$

$$\frac{\partial^2 \ln f(a_t)}{\partial c^2 \partial a_t} = \frac{1}{2\lambda^2 c^4}\left(\frac{\lambda}{\mu}\right)\left\{\left(\frac{a_t}{\mu}\right)^{\lambda - 1} - \left(\frac{a_t}{\mu}\right)^{-\lambda - 1}\right\},$$

we have

$$E\left\{\frac{1}{f(a_t)} \frac{\partial^2 f(a_t)}{\partial c^2 \partial a_t}\right\} = \frac{1}{2\lambda^2 c^4}\left(\frac{\lambda}{\mu}\right) E\left\{\left(\frac{a_t}{\mu}\right)^{\lambda - 1} - \left(\frac{a_t}{\mu}\right)^{-\lambda - 1}\right\}$$

$$+ E\left[\left(\frac{1}{2c^2} - \frac{1}{2\lambda^2 c^4}\left\{\left(\frac{a_t}{\mu}\right)^{\lambda} + \left(\frac{a_t}{\mu}\right)^{-\lambda} - 2\right\}\right)\left(\left(1 + \frac{\lambda}{2}\right)\frac{1}{a_t} + \right.\right.$$

$$\left.\left.\frac{1}{2(\lambda c)^2}\left(\frac{\lambda}{\mu}\right)\left\{\left(\frac{a_t}{\mu}\right)^{\lambda - 1} - \left(\frac{a_t}{\mu}\right)^{-\lambda - 1}\right\}\right)\right]$$

$$= \frac{1}{2\lambda^2 c^6 \mu}\left(\lambda^2 c^2 - c^2 + 2\right) \frac{1}{\lambda c} \sqrt{(2/\pi)} \exp\left(\frac{1}{\lambda^2 c^2}\right) \{K_{1/\lambda - 1/2}\left(\frac{1}{\lambda^2 c^2}\right)$$

$$- K_{1/\lambda + 3/2}\left(\frac{1}{\lambda^2 c^2}\right)\} - \frac{1}{4\lambda c^6 \mu} \frac{1}{\lambda c} \sqrt{(2/\pi)} \exp\left(\frac{1}{\lambda^2 c^2}\right).$$

$$\{K_{1/\lambda - 3/2}\left(\frac{1}{\lambda^2 c^2}\right) - K_{1/2 + 5/2}\left(\frac{1}{\lambda^2 c^2}\right)\}$$

$$+ \left(1 + \frac{\lambda}{2}\right)\left(1 + c^2\right) \frac{\lambda^2 c^2 + 2}{2\lambda^2 c^4 \mu}.$$

And from the following equations

$$E\{\frac{1}{f(a_t)}\ \frac{\partial^2 f(a_t)}{\partial \mu^2}\}=E\{\frac{\partial^2 \ln f(a_t)}{\partial \mu^2}\}+E\{(\frac{\partial \ln f(a_t)}{\partial \mu})^2\},$$

$$E\{\frac{\partial^2 \ln f(a_t)}{\partial \mu^2}\}=-\frac{\lambda}{2\mu^2}-\frac{\lambda}{2\mu^2}\frac{1}{(\lambda c)^2}(1+\lambda)E\{(\frac{a_t}{\mu})^{\lambda}-(\frac{a_t}{\mu})^{-\lambda}\},$$

$$E\{\frac{\partial \ln f(a_t)}{\partial \mu})^2\}=\frac{\lambda^2}{4\mu}+\frac{1}{4(\lambda c)^4}(\frac{\lambda}{\mu})^2E\{(\frac{a_t}{\mu})^{2\lambda}+(\frac{a_t}{\mu})^{-2\lambda}-2\}$$

$$+\frac{1}{2c^2\mu^2}E\{(\frac{a_t}{\mu})^{\lambda}-(\frac{a_t}{\mu})^{-\lambda}\},$$

$$E(\frac{a_t}{\mu})^{2\lambda}=1+(\lambda c)^2,\ \ E(\frac{a_t}{\mu})^{-2\lambda}=\{1+3(\lambda c)^2+3(\lambda c)^4\},$$

we have

$$E\{\frac{1}{f(a_t)}\ \frac{\partial^2 f(a_t)}{\partial \mu^2}\}=\frac{1}{c^2\mu^2}\{1+(\lambda c)^2\},$$

Further we obtain

$$E\{(\frac{\partial \ln f(a_t)}{\partial a_t})^2\}=(1+\frac{\lambda}{2})^2E(a_t^{-2})$$

$$+\frac{1}{4(\lambda c)^4}(\frac{\lambda}{\mu})^2E\{(\frac{a_t}{\mu})^{2(\lambda-1)}+(\frac{a_t}{\mu})^{-2(\lambda+1)}-(\frac{a_t}{\mu})^{-2}\}$$

$$+\frac{1}{\lambda c^2\mu^2}(1+\frac{\lambda}{2})E\{(\frac{a_t}{\mu})^{\lambda-2}-(\frac{a_t}{\mu})^{-\lambda-2}\}$$

$$=(1+\frac{\lambda}{2})^2\frac{1+3(\lambda c)^2+3(\lambda c)^4}{\mu^2}$$

$$+\frac{1}{4\lambda^2c^4\mu^2}\ \frac{1}{\lambda c}\sqrt{2/\pi}\exp(\frac{1}{\lambda^2c^2})\ \{K_{2/\lambda-3/2}(\frac{1}{\lambda^2c^2})$$

$$+K_{2/\lambda+5/2}(\frac{1}{\lambda^2c^2})\}$$

$$-\frac{(2+\lambda)(4+\lambda)}{2\mu^2}\ \frac{1}{\lambda c}\sqrt{2/\pi}\exp(\frac{1}{\lambda^2c^2})K_{2/\lambda+1/2}(\frac{1}{\lambda^2c^2}).$$

要　約

経済時系列においてAMRAモデル等を当てはめる際、イノヴェイションが非正規分布している場合が非常に多い。対数正規分布とカイ２乗分布の場合は調べられているが、その他の非正規分布に従う場合は知られていない。本論文ではイノヴェイションが逆正規分布（Inverse Gaussian distribution）に従う場合に条件付最尤推定量が存在するかどうか、存在するためにはどんな条件か必要かを求めた。なお対数正規分布と逆正規分布を特別な場合として含むべき等逆正規分布（power inverse Gaussian distribution）の場合について検討した。結果はパラメーターc^2の制限された値にたいして存在する。漸近的には条件付最尤推定量は存在し一致推定量となる。