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Integrability of $(\mathcal{A}, \mathcal{B})$-superharmonic functions

Takayori ONO

Abstract We investigate integrability of $(\mathcal{A}, \mathcal{B})$-superharmonic functions.

§1. Introduction and Preliminaries

Let Ω be a domain in $\mathbf{R}^N$ $(N \geq 2)$. As in [**MO1**], [**MO2**] and [**MO3**] we assume that $\mathcal{A} : \Omega \times \mathbf{R}^N \to \mathbf{R}^N$ and $\mathcal{B} : \Omega \times \mathbf{R} \to \mathbf{R}$ satisfy the following conditions for $1 < p < \infty$ and a *weight* w which is *p-admissible* in the sense of [**HKM**]:

(A.1) $x \mapsto \mathcal{A}(x, \xi)$ is measurable on Ω for every $\xi \in \mathbf{R}^N$ and $\xi \mapsto \mathcal{A}(x, \xi)$ is continuous for a.e. $x \in \Omega$;

(A.2) $\mathcal{A}(x, \xi) \cdot \xi \geq \alpha_1 w(x)|\xi|^p$ for all $\xi \in \mathbf{R}^N$ and a.e. $x \in \Omega$ with a constant $\alpha_1 > 0$;

(A.3) $|\mathcal{A}(x, \xi)| \leq \alpha_2 w(x)|\xi|^{p-1}$ for all $\xi \in \mathbf{R}^N$ and a.e. $x \in \Omega$ with a constant $\alpha_2 > 0$;

(A.4) $\big(\mathcal{A}(x, \xi_1) - \mathcal{A}(x, \xi_2)\big) \cdot \big(\xi_1 - \xi_2\big) > 0$ whenever $\xi_1, \xi_2 \in \mathbf{R}^N$, $\xi_1 \neq \xi_2$, for a.e. $x \in \Omega$;

(B.1) $x \mapsto \mathcal{B}(x, t)$ is measurable on Ω for every $t \in \mathbf{R}$ and $t \mapsto \mathcal{B}(x, t)$ is continuous for a.e. $x \in \Omega$;

(B.2) For any open set $D \Subset \Omega$, there is a constant $\alpha_3(D) \geq 0$ such that $|\mathcal{B}(x, t)| \leq \alpha_3(D)w(x)(|t|^{p-1} + 1)$ for all $t \in \mathbf{R}$ and a.e. $x \in D$;

(B.3) $t \mapsto \mathcal{B}(x, t)$ is nondecreasing on $\mathbf{R}$ for a.e. $x \in \Omega$.

We consider elliptic quasi-linear equations of the form

$$\text{(E)} \qquad -\operatorname{div} \mathcal{A}(x, \nabla u(x)) + \mathcal{B}(x, u(x)) = 0$$

on a domain Ω.

For the nonnegative measure $\mu : d\mu(x) = w(x)dx$ and an open set Ω, we consider the weighted Sobolev spaces $H^{1,p}(\Omega; \mu)$, $H_0^{1,p}(\Omega; \mu)$ and $H_{\mathrm{loc}}^{1,p}(\Omega; \mu)$ (see [**HKM**] for details).

Let Ω be an open subset of Ω. $u \in H_{\mathrm{loc}}^{1,p}(\Omega; \mu)$ is said to be a (weak) *solution* of (E) in Ω if

$$\int_\Omega \mathcal{A}(x, \nabla u) \cdot \nabla \varphi \, dx + \int_\Omega \mathcal{B}(x, u)\varphi \, dx = 0$$

for all $\varphi \in C_0^\infty(\Omega)$. $u \in H_{\mathrm{loc}}^{1,p}(\Omega; \mu)$ is said to be a *supersolution* (resp. *subsolution*) of (E) in Ω if

$$\int_\Omega \mathcal{A}(x, \nabla u) \cdot \nabla \varphi \, dx + \int_\Omega \mathcal{B}(x, u)\varphi \, dx \geq 0 \quad (\text{resp. } \leq 0)$$

for all nonnegative $\varphi \in C_0^\infty(\Omega)$.

A continuous solution of (E) in Ω is called $(\mathcal{A}, \mathcal{B})$-*harmonic* in Ω.

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A function $u : \Omega \to \mathbf{R} \cup \{\infty\}$ is said to be $(\mathcal{A}, \mathcal{B})$-*superharmonic* in Ω if it is lower semicontinuous, finite on a dense set in U and, for each open set $G \Subset \Omega$ and for $h \in C(\overline{G})$ which is $(\mathcal{A}, \mathcal{B})$-harmonic in G, $u \geq h$ on ∂G implies $u \geq h$ in G. $(\mathcal{A}, \mathcal{B})$-*subharmonic* functions are similarly defined.

Suppose that G is an open subset in Ω. Let u be a function in G such that $\min(u, k) \in H^{1,p}_{loc}(G; \mu)$ for all nonnegative integers k. Then we define

$$Du = \lim_{k \to \infty} \nabla \min(u, k).$$

The purpose of the present paper is to investigate integrability of $(\mathcal{A}, \mathcal{B})$-superharmonic functions. That is, we establish the following theorem

Theorem *Suppose that G is an open subset in Ω. If u is a nonnegative $(\mathcal{A}, \mathcal{B})$-superharmonic function, then $u \in L^{\gamma}_{loc}(G; \mu)$ and $Du \in L^{q(p-1)}_{loc}(G; \mu)$ whenever $0 < \gamma < \varkappa(p-1)$ and*

$$0 < q < \frac{\varkappa p}{\varkappa(p-1)+1},$$

where $\varkappa > 1$ is the exponent of the Sobolev inequality.

First, following the discussion in **[MZ]**, in which the unweighted case, namely the case $w = 1$, is treated, we will show the weak Harnack inequality for supersolutions of (E). Next, in the same manner as in **[HKM]**, in which the case $\mathcal{B} = 0$ in (E) is treated, we will discuss integrability of $(\mathcal{A}, \mathcal{B})$-superharmonic functions.

§2. Proof of Theorem

In this section, we will show the integrability of $(\mathcal{A}, \mathcal{B})$-superharmonic functions. For this, we first show the weak Harnack inequality for supersolutions of (E).

Let u be a nonnegative supersolution of (E) in Ω and $B(r) = B(x, r)$ be a ball with $B(r) \Subset \Omega$. We set $\bar{u} = u + r$. Thus, if $\eta \in C_0^{\infty}(B)$ is nonnegative, then $\varphi(x) = \eta^p \bar{u}^{\beta} \in H_0^{1,p}(B; \mu)$ for any real value of β . Moreover,

$$|\mathcal{B}(x, u)| \leq 2\alpha_3(B(r)) w \max(1, 1/r^{p-1}) \bar{u}^{p-1}.$$

We set $\alpha_3'(B(r)) = 2\alpha_3(B(r)) \max(1, 1/r^{p-1})$.

Lemma *Suppose that G is an open set with $G \Subset \Omega$ and $B(r) := B(x, r) \Subset G$. If u is a nonnegative supersolution of* (E) *in G, then, for any σ, $\tau \in (0, 1)$, there exists a constant $c = c(N, p, \alpha_1, \alpha_2, \alpha_3(G), r, \gamma, \sigma, \tau) > 0$ such that*

$$\left(\int_{B(x,\sigma r)} u^{\gamma} \, d\mu \right)^{1/\gamma} \leq c \left(\operatorname*{ess\,inf}_{B(x,\tau r)} u + r \right)$$

whenever $0 < \gamma < \varkappa(p-1)$, where $\varkappa > 1$ is the exponent in the Sobolev inequality.

Proof. Let $F : (0, \infty) \to \mathbf{R}$ be a smooth nonincreasing function, $\eta \in C_0^{\infty}(B(r))$ be nonnegative, $\overline{u} := u + r$ and $\varphi := F(u)\eta^p$. Then, since

$$\nabla \varphi = \eta^p F'(u) \nabla u + p F(u) \eta^{p-1} \nabla \eta,$$

we have

$$\int_{B(r)} \mathcal{A}(x, \nabla u) \cdot (F'(u)\eta^p \nabla u + pF(u)\eta^{p-1}\nabla\eta)\, dx + \int_{B(r)} \mathcal{B}(x,u)F(u)\eta^p dx \geq 0.$$

From (A.2), (A.3) and (B.2) it follows that

$$\begin{aligned}\alpha_1 \int_{B(r)} |\nabla u|^p |F'(u)| \eta^p\, d\mu \ &\leq\ p\alpha_2 \int_{B(r)} |\nabla u|^{p-1} |\nabla \eta| F(u) \eta^{p-1}\, d\mu \\ &\quad + \alpha_3(G) \int_{B(r)} (|u|^{p-1}+1) F(u)\eta^p\, d\mu.\end{aligned}$$

Setting $F(u) = \overline{u}^\beta$ $(\beta < 0)$, since $F'(u) = \beta u^{\beta-1}$, we have

$$\alpha_1|\beta| \int_{B(r)} |\nabla u|^p \overline{u}^{\beta-1}\eta^p\, d\mu \leq p\alpha_2 \int_{B(r)} |\nabla u|^{p-1}|\nabla\eta| \overline{u}^\beta \eta^{p-1}\, d\mu + \alpha_3' \int_{B(r)} \overline{u}^{p-1+\beta}\eta^p\, d\mu. \tag{1}$$

where $\alpha_3' = \alpha_3'(G)$. By Young's inequality, for any $\delta > 0$,

$$|\nabla u|^{p-1}|\nabla\eta|\overline{u}^\beta\eta^{p-1} \leq \delta^{-p/(p-1)}\frac{p-1}{p}|\nabla u|^p \overline{u}^{\beta-1}\eta^p + \delta^p \frac{1}{p}|\nabla \eta|^p \overline{u}^{p+\beta-1}.$$

Hence, by (1)

$$\begin{aligned}\alpha_1|\beta| \int_{B(r)} |\nabla u|^p \overline{u}^{\beta-1}\eta^p\, d\mu \ &\leq\ (p-1)\alpha_2\delta^{-p/(p-1)} \int_{B(r)} |\nabla u|^p \overline{u}^{\beta-1}\eta^p\, d\mu \\ &\quad + \alpha_2\delta^p \int_{B(r)} |\nabla\eta|^p \overline{u}^{p+\beta-1}\, d\mu + \alpha_3' \int_{B(r)} \overline{u}^{p-1+\beta}\eta^p\, d\mu.\end{aligned} \tag{2}$$

Now choose $\delta > 0$ so that

$$(p-1)\alpha_2\delta^{p/(p-1)} = \alpha_1 \frac{|\beta|}{2},$$

where $c = c\,(p, \alpha_1, \alpha_2)$. Thus, by (2) we have

$$\begin{aligned}\int_{B(r)} |\nabla u|^p \overline{u}^{\beta-1}\eta^p\, d\mu \ &\leq\ c\,|\beta|^{-p} \int_{B(r)} |\nabla\eta|^p \overline{u}^{p+\beta-1}\, d\mu \\ &\quad + |\beta|^{-1}\alpha_3' \int_{B(r)} \overline{u}^{p-1+\beta}\eta^p d\mu,\end{aligned} \tag{3}$$

where $c = c\,(p, \alpha_1, \alpha_2)$. Set

$$v := \begin{cases} \overline{u}^q & \text{if } pq = p+\beta-1,\ \beta \neq 1-p \\ \log \overline{u} & \text{if } \beta = 1-p. \end{cases}$$

If $\beta \neq 1-p$, by (3) we have

$$\begin{aligned}\int_{B(r)} |\nabla v|^p \eta^p\, d\mu \ &\leq\ 2c\,|q|^p|\beta|^{-p} \int_{B(r)} |\nabla\eta|^p v^p\, d\mu + |q|^p|\beta|^{-1}\alpha_3' \int_{B(r)} v^p\eta^p\, d\mu \\ &\leq\ c\,|q|^p(1+|\beta|^{-1})^p \int_{B(r)} (\eta^p + |\nabla\eta|^p) v^p\, d\mu,\end{aligned} \tag{4}$$

where $c = c(p, \alpha_1, \alpha_2, \alpha_3(G), r)$. If $\beta = 1 - p$, by (3) we have

$$\int_{B(r)} |\nabla v|^p \eta^p \, d\mu \le c \int_{B(r)} (\eta^p + |\nabla \eta|^p) \, d\mu, \tag{5}$$

where $c = c(p, \alpha_1, \alpha_2, \alpha_3(G), r)$. If $\beta \ne 1 - p$, the Sobolev inequality and (4) yield

$$\begin{aligned}
\left(\frac{1}{\mu(B(r))} \int_{B(r)} (\eta v)^{\varkappa p} \, d\mu \right)^{1/\varkappa p} &\le c_\mu r \left(\frac{1}{\mu(B(r))} \int_{B(r)} |\nabla(\eta v)|^p \, d\mu \right)^{1/p} \\
&\le 2 c_\mu r \{\mu(B(r))\}^{-1/p} \left(\int_{B(r)} (\eta^p |\nabla v|^p + |\nabla \eta|^p v^p) \, d\mu \right)^{1/p} \\
&\le c r \{\mu(B(r))\}^{-1/p} \left(|q|^p (1 + |\beta|^{-1})^p \int_{B(r)} (\eta^p + |\nabla \eta|^p) v^p \, d\mu + \int_{B(r)} |\nabla \eta|^p v^p \, d\mu \right)^{1/p}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\left(\int_{B(r)} (\eta v)^{\varkappa p} \, d\mu \right)^{1/\varkappa p} \le \; & c r \{\mu(B(r))\}^{\frac{1-\varkappa}{\varkappa p}} (1 + |q|)(1 + |\beta|^{-1}) \\
& \times \left(\int_{B(r)} (\eta^p + |\nabla \eta|^p) v^p \, d\mu \right)^{1/p},
\end{aligned} \tag{6}$$

where $c = c(p, \alpha_1, \alpha_2, \alpha_3, r, c_\mu)$.

Let $0 < h' < h \le r$ and $\eta \in C_0^\infty(B(h))$ be chosen so that $\eta = 1$ on $B(h')$, $0 \le \eta \le 1$ in $B(h)$ and $|\nabla \eta| \le 3(h - h')^{-1}$. Then, since $\eta \le 1 \le r(h - h')^{-1}$, (6) yields

$$\begin{aligned}
\left(\int_{B(h')} v^{\varkappa p} \, d\mu \right)^{1/\varkappa p} \le \; & c r \{\mu(B(h))\}^{\frac{1-\varkappa}{\varkappa p}} \{\max(3, r)\} (h - h')^{-1} (1 + |q|)(1 + |\beta|^{-1}) \\
& \times \left(\int_{B(h)} v^p \, d\mu \right)^{1/p}.
\end{aligned} \tag{7}$$

Set $s := pq = p + \beta - 1$. If $s > 0$, by (7) we have

$$\begin{aligned}
\left(\int_{B(h')} \overline{u}^{\varkappa s} \, d\mu \right)^{1/\varkappa s} \le \; & [c r \{\mu(B(h))\}^{\frac{1-\varkappa}{\varkappa p}} \{\max(3, r)\} (h - h')^{-1} (1 + s)(1 + |\beta|^{-1})]^{p/s} \\
& \times \left(\int_{B(h)} \overline{u}^s \, d\mu \right)^{1/s},
\end{aligned} \tag{8}$$

where $c = c(p, \alpha_1, \alpha_2, \alpha_3(G), h, c_\mu)$. If $s < 0$, we have $|\beta| > |1 - p| = p - 1$, so that $|\beta|^{-1} < (p - 1)^{-1}$. Hence, if $s < 0$, from (7) we obtain

$$\begin{aligned}
\left(\int_{B(h')} \overline{u}^{\varkappa s} \, d\mu \right)^{1/\varkappa s} \ge \; & [c r \{\mu(B(r))\}^{\frac{1-\varkappa}{\varkappa p}} \{\max(3, r)\} (h - h')^{-1} (1 - s)]^{p/s} \\
& \times \left(\int_{B(h)} \overline{u}^s \, d\mu \right)^{1/s}.
\end{aligned} \tag{9}$$

Let $0<\gamma<\varkappa(p-1)$. Fix $j=1,2,\ldots$. Setting $s_i=\varkappa^{i-j-1}\gamma \quad (i=1,2,\ldots,j+1)$, for any $i=0,1,2,\ldots,j$ we have

$$|\beta|=|s_i-(p-1)|=p-1-\frac{\gamma}{\varkappa}\varkappa^{i-j}\geq p-1-\frac{\gamma}{\varkappa}>0.$$

Hence, we have

$$|\beta|^{-1}<\left(p-1-\frac{\gamma}{\varkappa}\right)^{-1}.$$

Also, setting

$$h_i=r\{\sigma+2^{-i}(1-\sigma)\} \qquad \text{and} \qquad h_i'=h_{i+1},$$

since $s_i\leq\gamma$, by (8) we have

$$\left(\int_{B(h_{i+1})}\overline{u}^{\varkappa s_i}\,d\mu\right)^{1/\varkappa s_i}\leq [c\,r\,\{\mu(B(h_i))\}^{\frac{1-\varkappa}{\varkappa p}}\{\max(3,r)\}(h_i-h_{i+1})^{-1}(1+\gamma)]^{p/s_i}$$
$$\times\left(\int_{B(h_i)}\overline{u}^{s_i}\,d\mu\right)^{1/s_i},$$

where $c=c\,(p,\alpha_1,\alpha_2,\alpha_3(G),r,c_\mu,\gamma)$. Thus we obtain from the iteration

$$\left(\int_{B(\sigma r)}\overline{u}^{\gamma}\,d\mu\right)^{1/\gamma}\leq [c\,\{\mu(B(r))\}^{\frac{1-\varkappa}{\varkappa p}}\{\max(3,r)\}(1-\sigma)^{-1}(1+\gamma)]^{p\sum_{i=0}^{j}\frac{1}{s_i}}2^{p\sum_{i=0}^{j}\frac{i}{s_i}}$$
$$\times\left(\int_{B(r)}\overline{u}^{s_0}\,d\mu\right)^{1/s_0},$$

that is, for any $s_0\in\{\varkappa^{-j-1}\gamma : j=1,2,\ldots\}$

$$\left(\int_{B(\sigma r)}\overline{u}^{\gamma}\,d\mu\right)^{1/\gamma}\leq c\left(\int_{B(r)}\overline{u}^{s_0}\,d\mu\right)^{1/s_0}, \tag{10}$$

where $c=c\,(p,\alpha_1,\alpha_2,\alpha_3(G),r,c_\mu,\gamma,\sigma,\tau,s_0)$. By Hölder's inequality, the above inequality holds for any $s_0>0$.

Also, setting, for any $s_0>0$, $s_i=-\varkappa^i s_0$ and

$$h_i=r\{\tau+2^{-i}(1-\tau)\} \qquad \text{and} \qquad h_i'=h_{i+1},$$

by (9) we have

$$\left(\int_{B(h_{i+1})}\overline{u}^{\varkappa s_i}\,d\mu\right)^{1/\varkappa s_i}\geq [c\,r\,\{\mu(B(r))\}^{\frac{1-\varkappa}{\varkappa p}}\{\max(3,r)\}(h_i-h_{i+1})^{-1}\{1+(-s_i)\}]^{p/s_i}$$
$$\times\left(\int_{B(h_i)}\overline{u}^{s_i}\,d\mu\right)^{1/s_i}.$$

Since $1-s_i=1+\varkappa^i s_0\leq\varkappa^i+\varkappa^i s_0=(1+s_0)\varkappa^i$, again we obtain from the iteration

$$\left(\operatorname*{ess\,sup}_{B(\tau r)}\overline{u}^{-1}\right)^{-1}\geq [c\,\{\mu(B(r))\}^{\frac{1-\varkappa}{\varkappa p}}\{\max(3,r)\}(1-\tau)^{-1}]^{p\sum_{i=0}^{\infty}\frac{1}{s_i}}\{2(1+s_0)\varkappa\}^{p\sum_{i=0}^{\infty}\frac{i}{s_i}}$$
$$\times\left(\int_{B(r)}\overline{u}^{-s_0}\,d\mu\right)^{-1/s_0},$$

that is,

$$\text{ess}\inf_{B(\tau r)} \bar{u} \geq c \left(\int_{B(r)} \bar{u}^{-s_0}\, d\mu \right)^{-1/s_0}, \tag{11}$$

where $c = c\,(p, \alpha_1, \alpha_2, \alpha_3(G), r, c_\mu, \gamma, \sigma, \tau, s_0)$.

Finally, we show

$$\left(\int_{B(r)} \bar{u}^{s_0}\, d\mu \right)^{1/s_0} \leq c \left(\int_{B(r)} \bar{u}^{-s_0}\, d\mu \right)^{-1/s_0}. \tag{12}$$

Set $v = \log \bar{u}$. Let $B \subset B(r)$ be any ball of radius of h and $\eta \in C_0^\infty(2B)$ be so chosen that $\eta = 1$ on B, $0 \leq \eta \leq 1$ in $2B$ and $|\nabla \eta| \leq 3h^{-1}$. Then, (5) yields

$$\int_B |\nabla v|^p\, d\mu \leq c\, h^{-p} \mu(B)$$

where we have used $r/h \geq 1$ and the doubling property. By using Hölder's inequality and Poincaré inequality, we have

$$\int_B |v - v_B|\, d\mu \leq c\{\mu(B)\}^{(p-1)/p} h \left(\int_B |\nabla v|^p\, d\mu \right)^{1/p} \leq c\mu(B),$$

where $v_B = \frac{1}{\mu(B)} \int_B v\, d\mu$. Hence v satisfies the hypothesis of the John-Nirenberg lemma. Therefore, by the John-Nirenberg lemma, there are positive constants s_0 and c_0 such that

$$\left(\int_{B(r)} e^{s_0 v} d\mu \right) \left(\int_{B(r)} e^{-s_0 v} d\mu \right) \leq c_0 \{\mu(B(r))\}^2.$$

Hence we obtain (12), and by (10), (11) and (12) the proof is complete.

Now, using the above lemma, we give the proof of the theorem.

Proof of Theorem. Let G' be a polyhedoron such that $G' \Subset G$ and h_0 be the continuous solution of $-\text{div}\mathcal{A}(x, \nabla u) + \min\{\mathcal{B}(x, 0), 0\} = 0$ in G' with boundary values 0 on $\partial G'$. By the comparision principle, we see that $h_0 \geq 0$. For $k > 0$, let $u_k = \min(u, k + h_0)$. Since

$$-\text{div}\mathcal{A}(x, \nabla(k + h_0)) + \mathcal{B}(x, k + h_0) \geq -\text{div}\mathcal{A}(x, \nabla h_0) + \min\{\mathcal{B}(x, 0), 0\} = 0$$

and $k + h_0$ is continuous, $k + h_0$ is $(\mathcal{A}, \mathcal{B})$-superharmonic in G', and hence so is u_k. Moreover since h_0 is bounded, so is u_k, and hence u_k is a supersolution of (E) in G'.

Let $B = B(x, r)$ be a ball with $2B \subset G'$. By the lemma, we have

$$\left(\int_B u_k^\gamma\, d\mu \right)^{1/\gamma} \leq c\, \left(\text{ess}\inf_B u_k + r \right) \leq c\, \left(\text{ess}\inf_B u + r \right) < \infty$$

whenever $0 < \gamma < \varkappa(p-1)$ with a constant c independent of k. Hence, letting $k \to \infty$, we have $\int_B u^\gamma\, d\mu < \infty$.

Next, we show the integrability of Du. Let

$$0 < q < \frac{\varkappa p}{\varkappa(p-1)+1}.$$

Since $h_0 \geq 0$, $\min(u,k) = u = u_k$ on $\{u \leq k\}$, so that $\nabla \min(u,k) = \nabla u_k$ a.e. on $\{u \leq k\}$. Hence

$$\int_B |\nabla \min(u,k)|^{q(p-1)}\, d\mu = \int_{B\cap\{u\leq k\}} |\nabla \min(u,k)|^{q(p-1)}\, d\mu$$
$$= \int_{B\cap\{u\leq k\}} |\nabla u_k|^{q(p-1)}\, d\mu \leq \int_B |\nabla u_k|^{q(p-1)}\, d\mu.$$

Set $\overline{u_k} = u_k + 2r$. If $\varepsilon > 0$, by Hölder's inequality and (3) in the lemma we have

$$\int_B |\nabla u_k|^{q(p-1)}\, d\mu = \int_B |\nabla u_k|^{q(p-1)} \overline{u}_k^{-(1+\varepsilon)(p-1)q/p} \overline{u}_k^{(1+\varepsilon)(p-1)q/p}\, d\mu$$
$$\leq \left(\int_B |\nabla u_k|^p \overline{u}_k^{-1-\varepsilon} d\mu\right)^{(p-1)q/p} \left(\int_B \overline{u}_k^{(1+\varepsilon)(p-1)q/\{p-q(p-1)\}}\, d\mu\right)^{\{p-(p-1)q\}/p}$$
$$\leq c\left(\int_{2B} \overline{u}_k^{p-1-\varepsilon} d\mu\right)^{(p-1)q/p} \left(\int_B \overline{u}_k^{(1+\varepsilon)(p-1)q/\{p-q(p-1)\}}\, d\mu\right)^{\{p-(p-1)q\}/p}$$
$$\leq c\left(\int_{2B} (u+2r)^{p-1-\varepsilon} d\mu\right)^{(p-1)q/p} \left(\int_B (u+2r)^{(1+\varepsilon)(p-1)q/\{p-q(p-1)\}}\, d\mu\right)^{\{p-(p-1)q\}/p}.$$

Now choose ε so that $0 < \varepsilon < p-1$ and

$$\frac{(1+\varepsilon)(p-1)q}{p-q(p-1)} < \varkappa(p-1).$$

Thus, the integrability of u implies the integrability of Du.

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