CONSERVATION LAWS IN A BEHAVIOR OF A COMPLETE MONOPOLIST

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1. Introduction

Noether theorem (E. Noether [4]) concerning with symmetries of the action integral or its generalization (e.g. E. Bessel–Hagen[2]) with those up to divergence plays an effective role for discovering conservation laws from the Lagrangian or the Hamiltonian structures of considering problem. In this paper, the theorem is carried into a consideration of a complete monopolist with a cost function \( C(x) \) of \( x = D(p(t), \dot{p}(t)) \) \( (\dot{p} = dp/dt) \) where \( D(p(t), \dot{p}(t)) \) is his demand function for producing and selling a single good of price \( p = p(t) \) at time \( t \). Suppose that he behaves himself to maximize that profit over a period of time \([0,T]\):

\[
\int_0^T (xp - C(x)) dt = \int_0^T (D(\dot{p}, \dot{p}) \dot{p} - C(D(\dot{p}, \dot{p}))) dt.
\]

In the circumstances, Noether theorem is discussed to find some conserved quantities for the Lagrangian density

(1) \[ L(\dot{p}, \dot{p}) = D(\dot{p}, \dot{p}) \dot{p} - C(D(\dot{p}, \dot{p})) , \]

in particular, local conserved quantities for a quadratic Lagrangian density with respect to \( \dot{p} \), which can be reduced to that given by the following demand and the cost functions respectively (e.g. R.G.D.
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Allen[1])

(2) \[ D(p, \dot{p}) = ap + b\dot{p} + c (a, b, c : \text{const.}, ab \neq 0) \]
(3) \[ C(x) = \alpha x^2 + \beta x + \gamma (\alpha, \beta, \gamma : \text{const.}, \alpha \neq 0) \]

Moreover, another interesting illustration will be presented for a demand function \( D(p, \dot{p}) \) of homogeneous degree one with respect to the variables \( p \) and \( \dot{p} \). The Lagrangian density \( L(p, \dot{p}) \) can be generalized to the quadratic autonomous (time-independent) one for deriving more general local conserved quantities.

2. Noether theorem and conserved quantities

As well-known, Noether theorem[4] (or [2]) gives a procedure of discovering conserved quantities based on the symmetries (or symmetries up to divergence) of action integral of given Lagrangian density \( L \) under a group of transformations. Particularly for \( L=L(t, p, \dot{p}) \) and one-parameter group of infinitesimal transformations (\( \varepsilon \) denotes the parameter)

\[ \bar{t} = t + \varepsilon \tau(t, p), \]
\[ \bar{p} = p + \varepsilon \xi(t, p), \]

the theorem can be stated as follows:

Noether Theorem. For given Lagrangian density \( L=L(t, p, \dot{p}) \), assume that \( \tau = \tau(t, p), \xi = \xi(t, p) \) and \( \Phi = \Phi(t, p) \) satisfy the symmetry equation

\[
(4) \quad \frac{\partial L}{\partial t} \tau + \frac{\partial L}{\partial \dot{p}} \xi + \frac{\partial L}{\partial p} (\frac{\partial \xi}{\partial t} + \dot{p}(\frac{\partial \xi}{\partial p} - \frac{\partial \tau}{\partial t}) \frac{\partial \tau}{\partial p}) + L(\frac{\partial \tau}{\partial t} + \dot{p} \frac{\partial \tau}{\partial p}) = \frac{\partial \Phi}{\partial t} + \dot{p} \frac{\partial \Phi}{\partial p}.
\]
Then the quantity \( \Omega \):

\[
(5) \quad \Omega = (L - \dot{p} \frac{\partial L}{\partial \dot{p}} \tau + \frac{\partial L}{\partial p} \xi - \Phi
\]

is conserved, i.e., \( \Omega \) is constant along the trajectory \( p=p(t) \) of the Euler-Lagrange equation of \( L \) derived from the variational principle.

Suppose that the Lagrangian density \( L \) does not explicitly depend on the time \( t \), i.e., \( L=L(p,\dot{p}) \). Then, \( \tau = \text{const.}, \xi = 0 \) and \( \Phi = 0 \) satisfy (4). So that, in viewing (5), the quantity (H: Hamiltonian)

\[
(6) \quad -H = L - \dot{p} \frac{\partial L}{\partial \dot{p}}
\]

is conserved along the trajectory \( p=p(t) \). Since the Lagrangian density (1) lies on this case, we find immediately the conserved quantity

\[
-H = L - \dot{p} \frac{\partial L}{\partial \dot{p}} = pD - C - \dot{p}(\dot{p} \frac{\partial D}{\partial \dot{p}} - C' \frac{\partial D}{\partial \dot{p}})
\]

where \( C' = C'(x) = dC/dx \ (x = D) \).

3. Local conserved quantities

It seems to be difficult to obtain general solutions \( \tau, \xi \) and \( \Phi \) of (4) for the Lagrangian density (1). So, let the demand and the cost functions be of the forms (2) and (3) respectively. Then, (1) takes the form

\[
(7) \quad L(p,\dot{p}) = -b' \alpha \dot{p}^2 + b((1 - 2a \alpha) \dot{p} - (2c \alpha + \beta)) \dot{p} + a(1 - a \alpha) \dot{p}^2 + (c - (2c \alpha + \beta) a) \dot{p} - c^2 \alpha - c \beta - \gamma.
\]

So that, in general, we consider a quadratic Lagrangian density with
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respect to $$\dot{p}$$:

$$(8) \quad L(p, \dot{p}) = k\dot{p}^2 + f(\dot{p})\dot{p} + g(\dot{p}) \ (k: \text{const.}, \ k \neq 0),$$

where $$f(\dot{p})$$ is an arbitrary differentiable function and $$g(\dot{p})$$ is a second-order polynomial:

$$(9) \quad g(\dot{p}) = lp^2 + mp + n \ (l, m, n: \text{const.}, \ l \neq 0),$$

In this case, the symmetry equation (4) is reduced to a polynomial with respect to $$\dot{p}$$, whose left hand side is

$$-k\frac{\partial \tau}{\partial \dot{p}}\dot{p}^2 + k(2\frac{\partial \xi}{\partial \dot{p}} - \frac{\partial \tau}{\partial t})\dot{p}^2 + (f'\xi + 2k\frac{\partial \xi}{\partial t} + f\frac{\partial \xi}{\partial \dot{p}} + g\frac{\partial \tau}{\partial \dot{p}})\dot{p} + g'\xi + f\frac{\partial \xi}{\partial t} + g\frac{\partial \tau}{\partial t},$$

where $$f' = df/d\dot{p}$$ and $$g' = dg/d\dot{p}$$. So, it is satisfied for arbitrary $$\dot{p}$$ if and only if

$$\frac{\partial \tau}{\partial \dot{p}} = 0,$$

$$2\frac{\partial \xi}{\partial \dot{p}} - \frac{\partial \tau}{\partial t} = 0,$$

$$f'\xi + 2k\frac{\partial \xi}{\partial t} + f\frac{\partial \xi}{\partial \dot{p}} + g\frac{\partial \tau}{\partial \dot{p}} = \frac{\partial \Phi}{\partial \dot{p}},$$

$$g'\xi + f\frac{\partial \xi}{\partial t} + g\frac{\partial \tau}{\partial t} = \frac{\partial \Phi}{\partial t}.$$

In the equations, (10) implies $$\tau = \tau(t)$$, and then by putting $$\dot{t} = 2\varphi(t)$$, i.e.,

$$(14) \quad \tau = 2 \int \varphi(t)dt + K \ (K: \text{const.}) ,$$

(11) is rewritten as $$\partial \xi / \partial \dot{p} = \varphi$$, so that $$\xi$$ is of the form

$$(15) \quad \xi = \varphi(t)\dot{p} + \phi(t).$$
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Accordingly, (12) is rewritten as

\begin{equation}
\frac{\partial \Phi}{\partial p} = f' (\varphi p + \phi) + f \varphi + 2k (\varphi \varphi p + \phi) \\
= 2k \varphi p + 2k \phi + (fp)' \varphi + f' \phi;
\end{equation}

and (13) is also as follows, for which \( g \) of (9) is substituted:

\begin{equation}
\frac{\partial \Phi}{\partial t} = g' (\varphi p + \phi) + 2g \varphi + f (\varphi p + \phi) \\
= 4l \varphi \dot{p} + (3m \varphi + 2l \phi) p + 2n \varphi + m \phi + f (\varphi p + \phi).
\end{equation}

The equation (16) is integrated as

\[ \Phi = k \varphi \dot{p} + 2k \phi \dot{p} + f (\varphi p + \phi) + \phi (t), \]

and then differentiated with respect to \( t \):

\[ \frac{\partial \Phi}{\partial t} = k \varphi \dot{p} + 2k \phi \dot{p} + f (\varphi p + \phi) + \dot{\phi}, \]

which is to be identical to (17). So it follows that

\begin{align*}
(18) & \quad k \varphi = 4l \varphi, \\
(19) & \quad 2k \phi = 3m \varphi + 2l \phi, \\
& \quad \dot{\phi} = 2n \varphi + m \phi,
\end{align*}

Therefore, \( \varphi \) and \( \phi \) in \( \xi \) of (15) must satisfy (18) and (19), and \( \Phi \) be of the form

\begin{equation}
\Phi = k \varphi \dot{p} + 2k \phi \dot{p} + f (\varphi p + \phi) + 2n \int \varphi \, dt + m \int \phi \, dt.
\end{equation}

The solutions \( \varphi \) and \( \phi \) of (18) and (19) can be determined according to the signs of the following \( \sigma \). In fact by putting
\[ \lambda = \sqrt{1 - \sigma} \quad \text{where} \quad \sigma = \frac{t}{k}, \]

the solution of (18) is written as \((B_1, B_2: \text{const.})\)

(21a) \[ \varphi = B_1 e^{2\lambda t} + B_2 e^{-2\lambda t} \quad \text{if} \quad \sigma > 0, \]
(21b) \[ \varphi = B_1 \cos 2\lambda t + B_2 \sin 2\lambda t \quad \text{if} \quad \sigma < 0. \]

And the solution of (19) is written as a sum of the solution of \(k\dot{\varphi} = l \varphi:\)

\((A_1, A_2: \text{const.})\)

(22a) \[ \varphi_1 = A_1 e^{2\lambda t} + A_2 e^{-2\lambda t} \quad \text{if} \quad \sigma > 0, \]
(22b) \[ \varphi_1 = A_1 \cos 2\lambda t + A_2 \sin 2\lambda t \quad \text{if} \quad \sigma < 0. \]

and a particular solution \(\varphi_0\) of (19). It can be obtained by putting \(\varphi_0 = \mu \varphi\) \((\mu: \text{const.})\) and substituting for (19):

\[ 2k \mu \dot{\varphi} = (2\mu + 3m) \varphi, \]

which, together with (18), determines the constant \(\mu\) as \(\mu = m/2l\). Therefore the solution of (19) is given as

(23a) \[ \varphi = A_1 e^{2\lambda t} + A_2 e^{-2\lambda t} + \frac{m}{2l} (B_1 e^{2\lambda t} + B_2 e^{-2\lambda t}) \quad \text{if} \quad \sigma > 0, \]
(23b) \[ \varphi = A_1 \cos 2\lambda t + A_2 \sin 2\lambda t + \frac{m}{2l} (B_1 \cos 2\lambda t + B_2 \sin 2\lambda t) \quad \text{if} \quad \sigma < 0. \]

Thus for the Lagrangian density (8) with (9), the solutions \(\tau, \xi\) and \(\Phi\) are determined respectively as (14), (15) and (20); in which \(\varphi\) and \(\psi\) are given by (21a) and (23a), or (21b) and (23b), respectively. Consequently, the conserved quantity (5) leads to

\[ \Omega = (-kp^2 + g) \tau + (2kp + f) \xi - \Phi \]
\[ = K(-kp^2 + g) + 2k(\varphi p + \psi) \dot{p} - k \varphi \dot{p}^2 - 2k \psi \dot{p} + 2(-kp^2 + g - n) \int \varphi dt - m \int \psi dt, \]
in which, since \( K \) is an arbitrary constant, the following independent conserved quantities are observed:

\[
\Omega_1^0 = -k \dot{p}^2 + g - n = -k \dot{p}^2 + l \dot{p}^2 + m \dot{p}, \\
\Omega_2 = 2k(\varphi \dot{p} + \dot{\varphi}) - k \varphi \dot{\varphi} - 2k \dot{\varphi} \varphi + 2 \Omega_0^1 \int \varphi dt - m \int \dot{\varphi} dt.
\]

Alternatively, for the Lagrangian density (8) with (9), the conserved quantity \( \Omega_1^0 + n \) can be derived directly from (6), i.e.,

\[-H = L - \dot{p} \frac{\partial L}{\partial \dot{p}} = -k \dot{p}^2 + g = \Omega_1^0 + n.
\]

Moreover, by putting \( B_1 = B_2 = 0 \), i.e., \( \varphi = 0 \) and \( \dot{\varphi} = \dot{\psi} \), in the above \( \Omega_2 \), the conserved quantity is available:

\[
\Omega_2^0 = 2k \dot{\varphi} \dot{p} - 2k \varphi \dot{p} - m \int \dot{\varphi} dt,
\]

which leads, by (22a) or (22b), respectively to

\[
\Omega_2^0 = 2A_1(k \dot{p} - k \lambda \dot{p} - \frac{m}{2\lambda})e^{\cdot t} + 2A_2(k \dot{p} + k \lambda \dot{p} + \frac{m}{2\lambda})e^{-\cdot t} \quad \text{if} \quad \sigma > 0,
\]

\[
\Omega_2^0 = 2A_1(k \dot{p} \cos \lambda t + (k \dot{p} - \frac{m}{2\lambda})\sin \lambda t) + 2A_2(k \dot{p} \sin \lambda t - (k \dot{p} - \frac{m}{2\lambda})\cos \lambda t) \quad \text{if} \quad \sigma < 0,
\]

where the constant in \( \Omega_2^0 \) is omitted for the brevity. Since \( A_1 \) and \( A_2 \) are arbitrary constants, the following independent conserved quantities are observed in the above \( \Omega_2^0 \):

\[
\Omega_2^{01} = (k \dot{p} - k \lambda \dot{p} - \frac{m}{2\lambda})e^{\cdot t},
\]

\[
\Omega_2^{02} = (k \dot{p} + k \lambda \dot{p} + \frac{m}{2\lambda})e^{-\cdot t} \quad \text{if} \quad \sigma > 0;
\]

\[
\Omega_2^{01} = k \dot{p} \cos \lambda t + (k \lambda \dot{p} - \frac{m}{2\lambda})\sin \lambda t,
\]

\[
\Omega_2^{02} = k \dot{p} \sin \lambda t - (k \lambda \dot{p} - \frac{m}{2\lambda})\cos \lambda t \quad \text{if} \quad \sigma < 0.
\]
Here, in viewing \( \lambda^2 = \pm \sigma = \pm l/k \) for \( \sigma \geq 0 \) (the signs correspond respectively), note that \( \Omega^0_{21} \) and \( \Omega^0_{22} \) of (26a) and (27a), or (26b) and (27b), have relation to \( \Omega^0_{21} \) of (24):

\[
\begin{align*}
(28a) \quad & \Omega^0_{21} \Omega^0_{22} = -k \Omega^0_1 - \frac{m^2}{4\sigma} \quad \text{if } \sigma > 0, \\
(28b) \quad & (\Omega^0_{21})^2 + (\Omega^0_{22})^2 = -k \Omega^0_1 - \frac{m^2}{4\sigma} \quad \text{if } \sigma < 0.
\end{align*}
\]

The conserved quantity \( \Omega^0_2 \) of (25) is differentiated as

\[
\Omega^0_2 = 2k(\varphi_1 \ddot{p} - \ddot{\varphi}_1 p) - m \varphi_1 = 0,
\]

which, in terms of \( k \dot{\varphi}_1 = l \varphi_1 \), leads to

\[
2k \ddot{p} - 2l \dot{p} = m;
\]

whose solution \( p=\dot{p}(t) \) is given by its particular solution \( p_0 = -m/2l \) and the solution \( p_1 = \varphi_1 \) of \( k \dot{p} - lp = 0 \), i.e.,

\[
\begin{align*}
(29a) \quad & p(t) = A_1 e^{\lambda t} + A_2 e^{-\lambda t} - \frac{m}{2l} \quad \text{if } \sigma > 0, \\
(29b) \quad & p(t) = A_1 \cos \lambda t + A_2 \sin \lambda t - \frac{m}{2l} \quad \text{if } \sigma < 0.
\end{align*}
\]

The trajectory (29a) or (29b) can be derived also by elimination \( \dot{p} \) in the independent conserved quantities (26a) and (27a), or (26b) and (27b). Thus we have the following results.

**Theorem 1.** For the Lagrangian density \( L \) of (8) with (9), the quantities \( \Omega^0_1 \) of (24), \( \Omega^0_{31} \) and \( \Omega^0_{32} \) of (26a) and (27a), or of (26b) and (27b), are conserved along the trajectory \( p=\dot{p}(t) \) of the Euler-Lagrange equation of \( L \) derived from the variational principle. In the quantities \( \Omega^0_1, \Omega^0_{31} \) and \( \Omega^0_{32} \), two of them are independent but three are related as (28a) or (28b). Moreover, the
trajectory is determined completely as (29a) or (29b).

Remark. It can be determined a class of autonomous (time-independent) Lagrangian densities equivalent to \( L(p, \dot{p}) \) in Theorem 1 provided with the Euler-Lagrange equation

\[
2k\ddot{p} - 2l\dot{p} - m = 0.
\]

In fact a new one \( \bar{L}(p, \dot{p}) \) is equivalent to \( L(p, \dot{p}) \) (\( \bar{L} \) is subordinate to \( L[3] \)) if and only if, for the left hand side of the Euler-Lagrange equation of \( \bar{L}(p, \dot{p}) \), there exists a function \( h(p, \dot{p}) \) such that

\[
\dot{p} \frac{\partial^2 \bar{L}}{\partial p^2} + \ddot{p} \frac{\partial^2 \bar{L}}{\partial \dot{p} \partial p} - \frac{\partial \bar{L}}{\partial p} = h(2k\ddot{p} - 2l\dot{p} - m), \text{i.e.,}
\]

\[
\frac{\partial^2 \bar{L}}{\partial \dot{p}^2} = 2kh, \quad \dot{p} \frac{\partial^2 \bar{L}}{\partial \dot{p} \partial p} - \frac{\partial \bar{L}}{\partial p} = -(2lk + m)h.
\]

Consequently, by eliminating \( h \) and then multiplying \( \dot{p} \), it follows that

\[
(2lp + m) \frac{\partial H}{\partial \dot{p}} + 2k\dot{p} \frac{\partial H}{\partial p} = 0,
\]

where \(-H = \bar{L} - \dot{p} \partial \bar{L} / \partial \dot{p}\). So the subsidiary equation \( dp/(2lp + m) = dp/2k\dot{p} \) is integrated:

\[
\omega = k\ddot{p} - l\dot{p} - m\dot{p} = \text{const.},
\]

to put \( H = \dot{p} \partial \bar{L} / \partial \dot{p} - \bar{L} \) as

\[
\dot{p} \frac{\partial \bar{L}}{\partial \dot{p}} - \bar{L} = \Psi(\omega), \quad \text{i.e.,} \quad \frac{\partial}{\partial \dot{p}}(\frac{\bar{L}}{\dot{p}}) = \frac{\Psi(\omega)}{\dot{p}^2}.
\]

Therefore \( L(p, \dot{p}) \) is determined completely as
\( \bar{L}(p, \dot{p}) = \dot{p} \int \frac{\Psi(\omega)}{\dot{p}^2} d\dot{p} + f(p) \dot{p}, \)

which will give rise to the same conserved quantities in Theorem 1. Of course, since for \( \Psi(\omega) = \omega - n \) (\( n \): const.),

\[ \int \frac{\Psi(\omega)}{\dot{p}^2} d\dot{p} = k \dot{p} + \frac{l \dot{p}^2 + m \dot{p} + n}{\dot{p}} + g(p), \]

the class of equivalent Lagrangian densities of the form \( \bar{L}(p, \dot{p}) \) contains the original one of the form \( L(p, \dot{p}) \).

Theorem 1 is now applied to the Lagrangian density (1) given by the cost function (3):

\[ L(p, \dot{p}) = D(p, \dot{p}) \dot{p} - C(D(p, \dot{p})) = -\alpha D^2 + (p - \beta)D - \gamma, \]

so that the demand function is to be

\[ D(p, \dot{p}) = \frac{p - \beta \pm \sqrt{(p - \beta)^2 - 4\alpha (L + \gamma)}}{2\alpha}, \]

For the particular Lagrangian density \( L \) of the form (7), since

\[ (p - \beta)^2 - 4\alpha (L + \gamma) = (2b \alpha p + (2a \alpha - 1) p + 2c \alpha + \beta)^2, \]

\( D(p, \dot{p}) \) is determined as

\[ D(p, \dot{p}) = ap + bp + c \text{ or } D(p, \dot{p}) = \frac{1 - a \alpha}{\alpha} p - bp - \frac{\beta + c \alpha}{\alpha}, \]

in which appears the demand function of (2). And, if the cost function (3) is reduced to

\[ C(x) = \alpha x^2 + \gamma \quad (\alpha, \gamma: \text{const.}, \alpha \neq 0), \]

\[ -10 - \]
i.e., $\beta = 0$ in (3), a demand function of homogeneous degree one with respect to the variables $p$ and $\dot{p}$ can be derived, through (30), from a Lagrangian density

$$L(p, \dot{p}) = -\alpha p \dot{p} - \alpha \tau p \dot{p} - \frac{4\alpha^2 \kappa - 1}{4\alpha} \dot{p}^2 - \gamma,$$

where $\rho$, $\tau$ and $\kappa$ are some constants with $\rho \neq 0$, $4\alpha^2 \kappa \neq 1$. In fact, since

$$(p - \beta)^2 - 4\alpha (L + \gamma) = 4\alpha^2 (\rho \dot{p}^2 + \tau p \dot{p} + \kappa \dot{p}^2),$$

$D(p, \dot{p})$ is determined as

$$D(p, \dot{p}) = \frac{p}{2\alpha} \pm \sqrt{\rho \dot{p}^2 + \tau p \dot{p} + \kappa \dot{p}^2} \quad (\rho, \tau, \kappa: \text{const.}).$$

The above two cases for (3) and (31), or (3)' and (32), have the Lagrangian density of the form (8) with (9), where

$$k = -b^2 \alpha, \quad f(p) = b(1 - 2a \alpha)p - b(2c \alpha + \beta),$$

$$l = a(1 - a \alpha), \quad m = c - (2c \alpha + \beta)a, \quad n = -c^2 \alpha - c \beta - \gamma; \quad \text{or}$$

$$k = \alpha p, \quad f(p) = -\alpha \tau p, \quad l = \frac{4\alpha^2 \kappa - 1}{4\alpha}, \quad m = 0, \quad n = -\gamma,$$

respectively. Here note that $k \neq 0$ and $l \neq 0$. In these circumstances, Theorem 1 can be reviewed as follows.

**Theorem 2.** Let a complete monopolist have the cost and the demand functions of the forms (i): (3) and (31) (which includes (2)) with $ab \alpha \neq 0$, $a \alpha \neq 1$; or (ii): (3)' and (32) with $\alpha \rho \neq 0$, $4\alpha^2 \kappa \neq 1$, respectively. Then, in his behavior of maximizing the profit over a period of time, there exist the following three conserved quantities.
For the case (i), $\Xi_1 = \Omega_0^0/b^2A$ and $\Xi_{21} = -\Omega_{21}^0/\beta^2A$, $\Xi_{22} = -\Omega_{22}^0/b^2A$:

\[
\Xi_1 = \dot{p} - \sigma \dot{p}^2 + \frac{c-(2cA + \beta)a}{b^2A}p,
\]
\[
\Xi_{21} = (\dot{p} - \sigma \dot{p} \sqrt{\frac{c-(2cA + \beta)a}{b^2A}})e^{\sigma t},
\]
\[
\Xi_{22} = (\dot{p} + \sigma \dot{p} \sqrt{\frac{c-(2cA + \beta)a}{b^2A}})e^{-\sigma t}, \quad \text{if } \sigma > 0,
\]
\[
\Xi_1 = \dot{p} - \sigma \dot{p}^2 + \frac{c-(2cA + \beta)a}{b^2A}p,
\]
\[
\Xi_{21} = p \cos \sqrt{-\sigma}t + (\sqrt{-\sigma} \dot{p} + \frac{c-(2cA + \beta)a}{b^2A} \sqrt{-\sigma}) \sin \sqrt{-\sigma}t,
\]
\[
\Xi_{22} = p \sin \sqrt{-\sigma}t - (\sqrt{-\sigma} \dot{p} + \frac{c-(2cA + \beta)a}{b^2A} \sqrt{-\sigma}) \cos \sqrt{-\sigma}t, \quad \text{if } \sigma < 0,
\]

where $\sigma = \frac{a(aA - 1)}{b^2A}$.

For the case (ii), $\Xi_1 = \Omega_0^0/A\rho$ and $\Xi_{21} = -\Omega_{21}^0/A\rho$, $\Xi_{22} = -\Omega_{22}^0/A\rho$:

\[
\Xi_1 = \dot{p} - \sigma \dot{p},
\]
\[
\Xi_{21} = (\dot{p} - \sigma \dot{p})e^{\sigma t},
\]
\[
\Xi_{22} = (\dot{p} - \sigma \dot{p})e^{-\sigma t} \quad \text{if } \sigma > 0;
\]
\[
\Xi_1 = \dot{p} - \sigma \dot{p},
\]
\[
\Xi_{21} = \dot{p} \cos \sqrt{-\sigma}t + \sqrt{-\sigma} \dot{p} \sin \sqrt{-\sigma}t,
\]
\[
\Xi_{22} = \dot{p} \sin \sqrt{-\sigma}t - \sqrt{-\sigma} \dot{p} \cos \sqrt{-\sigma}t \quad \text{if } \sigma < 0,
\]

where $\sigma = \frac{4A^2 - 1}{4A^2 \rho}$.

In the quantities $\Xi_1$, $\Xi_{21}$ and $\Xi_{22}$, two of them are independent but three are related as

\[
\Xi_{21} \Xi_{22} = \Xi_1 - \frac{m^2}{4b^4A^2 \sigma} \quad \text{if } \sigma > 0,
\]
\[
\Xi_{21}^2 + \Xi_{22}^2 = \Xi_1 - \frac{m^2}{4b^2A^2 \sigma} \quad \text{if } \sigma < 0,
\]
where for the case (i): \( m = c - (2c\alpha + \beta)a \), or for the case (ii): \( m = 0 \). Moreover, the trajectory \( p = p(t) \) for the maximizing problem is determined completely as

\[
\begin{align*}
p(t) &= A_1 e^{\alpha t} + A_2 e^{-\alpha t} + p_0 & \text{if } \alpha > 0, \\
p(t) &= A_1 \cos \sqrt{-\sigma} t + A_2 \sin \sqrt{-\sigma} t + p_0 & \text{if } \sigma < 0,
\end{align*}
\]

where, for the case (i): \( p_0 = (c - (2c\alpha + \beta)a)/2a(a\alpha - 1) \), or for the case (ii): \( p_0 = 0 \).

4. More general local conserved quantities

A further consideration of Noether theorem is continued for the quadratic autonomous Lagrangian density with respect to \( \dot{q} \):

\[
L(\dot{p}, \dot{q}) = k(p)\dot{q}^2 + f(p)\dot{p} + g(p),
\]

where \( k(p) (k(p) > 0 \text{ or } k(p) < 0) \), \( f(p) \) and \( g(p) \) are arbitrary differentiable functions. In this generalization of (8), \( k' \dot{\xi} \dot{p} \) \((k' = dk/dp)\) appears in the term \((\partial L/\partial \dot{p}) \dot{\xi}\) of (4). Accordingly, in the system of (10), ..., (13), the equation (11) turns into

\[
k' \dot{\xi} + k(2 \frac{\partial \xi}{\partial \dot{p}} - \frac{\partial \tau}{\partial t}) = 0, \text{ i.e.,}
\]

\[
\frac{\partial (\sqrt{|k|} \dot{\xi})}{\partial \dot{p}} = \frac{\sqrt{|k|}}{2} \frac{\partial \tau}{\partial t},
\]

So that, particularly by putting \( \varphi = 0 \) in (14), i.e.,

\[
\tau = K \quad (K: \text{const.}),
\]

\( \xi \) is determined as
(35) \[ \xi = \frac{\phi(t)}{u}, \text{ where } u = \sqrt{|k|}. \]

Together with (34), the derivatives (12) and (13) of \( \Phi \) are substituted for

\[ \frac{\partial}{\partial t} \left( \frac{\partial \Phi}{\partial \rho} \right) = \frac{\partial}{\partial \rho} \left( \frac{\partial \Phi}{\partial t} \right), \]

to derive the condition

\[ 2k \frac{\partial^2 \xi}{\partial t^2} = g'' \xi + g' \frac{\partial \xi}{\partial \rho}, \]

which is, by (35), reduced to

\[ \frac{\phi'}{\phi} = \frac{2kg'' - k'g'}{4k^2}. \]

Since the left hand side of the identity is a function of \( t \), while the right one is of \( \rho \), both are constant:

(36) \[ \frac{\phi'}{\phi} = \frac{2kg'' - k'g'}{4k^2} = \sigma \quad (\sigma : \text{const.}) \]

So that (in what follows, the signs \( \pm \), also \( \mp \), correspond respectively to \( k \geq 0 \))

\[ \frac{\partial}{\partial \rho} \left( \frac{g'}{u} \right) = \pm 2\sigma u, \]

which is integrated as (note that \( g'/u \) is a function of only \( \rho \))

\[ \frac{g'}{u} = \pm 2\sigma \int u \, d\rho + \mu, \quad \text{i.e.,} \]

(37) \[ g' = \pm 2\sigma u \int u \, d\rho + \mu u \quad (\mu : \text{const.}). \]

Therefore \( g \) must be of the form

(38) \[ g = \pm \sigma (\int u \, d\rho)^2 + \mu \int u \, d\rho + \nu \quad (\sigma, \mu, \nu : \text{const.}). \]
For all such functions $g$, by putting
\[ \lambda = \sqrt{|\sigma|}, \]
the solution of $\phi$ in (36) is written as ($A_1$, $A_2$: const.)

(39a) \[ \phi = A_1 e^{it} + A_2 e^{-it} \quad \text{if } \sigma > 0, \]

(39b) \[ \phi = A_1 \cos \lambda t + A_2 \sin \lambda t \quad \text{if } \sigma < 0, \]

(39c) \[ \phi = A_1 t + A_2 \quad \text{if } \sigma = 0. \]

Now the function $\Phi$ can be determined by substituting (34) and (35), for (12) or (13), respectively:
\begin{align*}
\frac{\partial \Phi}{\partial p} &= \frac{\partial (f \xi)}{\partial p} \pm 2 \phi u, \\
\frac{\partial \Phi}{\partial t} &= g' \xi + f \frac{\partial \xi}{\partial t}.
\end{align*}

In fact, the first equation is integrated:
\[ \Phi = f \xi \pm 2 \phi \int u dp + \phi (t), \]
and then differentiated with respect to $t$:
\[ \frac{\partial \Phi}{\partial t} = f \frac{\partial \xi}{\partial t} \pm 2 \phi \int u dp + \phi, \]
which is identical to the second one. So that
\[ \phi = g' \xi \mp 2 \phi \int u dp, \]
for which (35), (36) and (37) are substituted to lead
\begin{align*}
\phi &= (\pm \sigma \int u dp + \mu) \phi \mp 2 \phi \int u dp \\
&= \pm 2(\sigma \phi - \phi) \int u dp + \mu \phi = \mu \phi, \quad \text{i.e.,} \\
\phi &= \mu \int \phi dt.
\end{align*}
Therefore the final form of $\Phi$ is

$$
(40) \quad \Phi = \frac{f \phi}{u} \pm 2 \phi \int u dp + \mu \int \phi dt.
$$

Thus, for the Lagrangian density satisfying (38), a system of solution $\gamma, \xi$ and $\Phi$ are determined respectively as (34), (35) and (40). For the system, the terms in (5) lead respectively to

$$
(41) \quad \Lambda_1 = L - \frac{b \partial L}{\partial p} = - kp^2 + g,
$$

$$
(42) \quad \Lambda_2 = \frac{\partial L}{\partial p} \xi - \Phi = \pm 2 \psi u p \mp 2 \phi \int u dp - \mu \int \phi dt - \nu.
$$

In $\Lambda_2$ by substituting (39a), (39b) or (39c), the following independent conserved quantities are observed:

$$
(43a) \quad \Lambda_{21} = (up - \lambda \int u dp \pm \frac{\mu}{2 \lambda}) e^{it},
$$

$$
(44a) \quad \Lambda_{22} = (up + \lambda \int u dp \pm \frac{\mu}{2 \lambda}) e^{-it}, \quad \text{if } \sigma > 0;
$$

$$
(43b) \quad \Lambda_{21} = u p \cos \lambda t + (\lambda \int u dp \mp \frac{\mu}{2 \lambda}) \sin \lambda t,
$$

$$
(44b) \quad \Lambda_{22} = u p \sin \lambda t - (\lambda \int u dp \pm \frac{\mu}{2 \lambda}) \cos \lambda t \quad \text{if } \sigma < 0;
$$

$$
(43c) \quad \Lambda_{21} = t u p - \int u dp \mp \frac{\mu}{4} t^2
$$

$$
(44c) \quad \Lambda_{22} = u p \mp \frac{\mu}{2} t \quad \text{if } \sigma = 0.
$$

Here, in the quantities $\Lambda_1, \Lambda_{21}$ and $\Lambda_{22}$, the following relations are derived by virtue of (38):

$$
(45a) \quad \Lambda_{21} \Lambda_{22} = \mp \Lambda_1 - \frac{\mu^2}{4 \sigma} \pm \nu \quad \text{if } \sigma > 0;
$$

$$
(45b) \quad \Lambda_{21}^2 + \Lambda_{22}^2 = \mp \Lambda_1 - \frac{\mu^2}{4 \sigma} \pm \nu \quad \text{if } \sigma < 0;
$$

$$
(45c) \quad \Lambda_{22} \pm \mu \Lambda_{21} = \mp \Lambda_1 \pm \nu \quad \text{if } \sigma = 0.
$$
The conserved quantity $\Lambda_2$ of (42) is differentiated:

$$ \pm \dot{\Lambda}_2 = 2 \Phi u' \dot{p} + 2 \Phi u \dot{p} - 2 \Phi \int u dp = \pm \mu \phi = 0, $$
in which $\phi = \sigma \phi$ is used to derive

$$ u' \dot{p} + u \dot{p} = \sigma \int u dp \pm \frac{\mu}{2}, \quad \text{i.e.,} \quad G = \sigma G \pm \frac{\mu}{2} \quad \text{where} \quad G = \int u dp. $$

The solution of $G$ is given, if $\sigma \neq 0$, as a sum of a particular solution $G_0 = \mp \mu / 2 \sigma$ and the solution of $G = \sigma G$, i.e., $G = \phi$ (while, if $\sigma = 0$, $G = \pm \mu / 2$ is directly integrated):

$$ (46a) \quad \int u dp = A_1 e^{\lambda t} + A_2 e^{-\lambda t} \mp \frac{\mu}{2 \sigma} \quad \text{if} \quad \sigma > 0, $$

$$ (46b) \quad \int u dp = A_1 \cos \lambda t + A_2 \sin \lambda t \mp \frac{\mu}{2 \sigma} \quad \text{if} \quad \sigma < 0, $$

$$ (46c) \quad \int u dp = \pm \frac{\mu}{4} t^2 + A_1 t + A_2 \quad \text{if} \quad \sigma = 0. $$

The above relations can be derived also by elimination $\dot{p}$ in the independent conserved quantities (43a) and (44a), or (43b) and (44b), or (43c) and (44c), respectively. Thus Theorem 1 is generalized as follows.

**Theorem 3.** For the quadratic autonomous Lagrangian density $L$ of (33) with respect to $\dot{p}$, satisfying the identity (38); the quantities $\Lambda_1$ of (41), $\Lambda_2$ and $\Lambda_3$ of (43a) and (44a), or (43b) and (44b), or (43c) and (44c), are conserved along the trajectory $p = p(t)$ of the Euler–Lagrange equation of $L$ derived from the variational principle. In the quantities $\Lambda_1$, $\Lambda_2$ and $\Lambda_3$, two of them are independent but three are related as (45a) or (45b) or (45c). Moreover, the trajectory satisfies the relation (46a) or (46b) or (46c). In the identities, the signs $\pm$, also $\mp$, correspond respectively to $k \geq 0$. 
Remark. Since the function $g$ of (9) is written as ($k$: const.)

$$g = \pm \frac{l}{k}(\int u dp)^2 + \frac{m}{\sqrt{|k|}} \int u dp + n,$$

where $\int u dp = \int \sqrt{|k|} dp = \sqrt{|k|} \rho$,

Theorem 3 is reduced to Theorem 1 by putting $\sigma = l/k, \mu = m/\sqrt{|k|}$ and $\nu = n$.

We are now in the position to generalize the demand function $x = D(p, \dot{p})$ of (31) as

$$D(p, \dot{p}) = \eta(p) + \zeta(p) \dot{p} \text{ or } D(p, \dot{p}) = -\eta(p) - \zeta(p) \dot{p} + \frac{p - \beta}{\alpha},$$

while leave the cost function of (3) as it is. Then the Lagrangian density $L(p, \dot{p}) = x \dot{p} - C(x)$ takes the form of (33), where

$$k(p) = -\alpha \zeta^2, \quad f(p) = -\zeta(2\alpha n - \rho + \beta), \quad g = -\eta(\alpha \eta - \rho + \beta) - \gamma.$$

So that, by substituting

$$u = \sqrt{|k|} = \sqrt{|\alpha|} |\zeta|, \text{ i.e., } \int u dp = \sqrt{|\alpha|} \int |\zeta| dp,$$

for (38), it follows the identity

$$\eta(\alpha \eta - \rho + \beta) + \gamma = \alpha \sigma (\int |\zeta| dp)^2 - \mu \sqrt{|\alpha|} \int |\zeta| dp - \nu.$$

As an illustrative example of (47), a demand function

$$(47)' \quad D(p, \dot{p}) = \frac{a}{p} + \frac{bb}{p^2} \text{ or } D(p, \dot{p}) = -\frac{a}{p} - \frac{bb}{p^2} + \frac{p - \beta}{\alpha}$$

$(a, b$: const., $ab \neq 0)$,
i.e., \( \eta = a/p \) and \( \zeta = b/p^2 \) satisfy (48) with the constants \( \sigma, \mu \) and \( \nu \):

\[
\sigma = \frac{a^2}{b^2}, \quad \mu = \frac{a \beta}{|b| \sqrt{|\alpha|}}, \quad \nu = a - \gamma.
\]

In this case, since \( \sigma > 0 \), the characters: \( \lambda = \sqrt{\sigma}, \mu \) and

\[
k = -\frac{b' \alpha}{p'b'}, \quad g = -\frac{a^2 \alpha}{p'} - \frac{a \beta}{p} + a - \gamma,
\]

\[
u = \sqrt{|k|} = \frac{|b| \sqrt{|\alpha|}}{p'}, \quad \int udP = -\frac{|b| \sqrt{|\alpha|}}{p'}
\]
take part in (41), (43a), (44a), (45a) and (46a). Then the resulting identities are rearranged as follows.

**Theorem 4.** Theorem 3 can be applied to the lagrangian density

\[ L(p, b) = xp - C(x) \]

given by the cost and the demand functions (3) and (47) satisfying the identity (48), respectively. By way of an example, let a complete monopolist have the cost and the demand functions of the form (3) and (47)', respectively. Then, in his behavior of maximizing the profit over a period of time, there exist the conserved quantities \( \Xi_1 = (\Lambda_1 - a + \gamma)/\alpha \), \( \Xi_{21} = \Lambda_{21}/\sqrt{|\alpha|} \) and \( \Xi_{22} = \Lambda_{22}/\sqrt{|\alpha|} \)

\[
\Xi_1 = \frac{b'p^2}{p^4} - \frac{a^2}{p^2} - \frac{a \beta}{\alpha P}, \\
\Xi_{21} = \frac{|b|}{p^2} + \frac{|a|}{p} + \frac{|a| \beta}{2a \alpha} \ e^{\frac{a|}{b'}}t, \\
\Xi_{22} = \frac{|b|}{p^2} + \frac{|a|}{p} + \frac{|a| \beta}{2a \alpha} \ e^{\frac{a|}{b'}}t.
\]

In the quantities \( \Xi_1, \Xi_{21} \) and \( \Xi_{22} \), two of them are independent but three are related as

\[
\Xi_{21} \Xi_{22} = \Xi_1 - \frac{\beta^2}{4 \alpha^2}.
\]
Moreover, the trajectory $p=p(t)$ for the maximizing problem is determined completely as

$$p(t) = \frac{2a \alpha}{A_1 e^{\frac{\beta}{A} t} + A_2 e^{-\frac{\beta}{A} t} - \beta}.$$

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